Gamma & Related Functions (c.f. A+S Ch. 6)

1. $\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$; $\mathrm{Re}\, z > 0$.

2. For $z = n \in \{0, 1, 2 \ldots\}$

$$\Gamma(n+1) = n! = n(n-1)\cdots 1 .$$

3. Factorial Property: $z\Gamma(z) = \Gamma(z+1)$

4. Reflection Property: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$

N.B. $\Gamma(z)$ has poles at $z = 0, -1, -2, -3, \ldots$

5. Special Values: $\Gamma(1) = 0! = 1$, $\Gamma(2) = 1! = 1$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

6. Relation to Gaussian Integrals

$$\int_0^\infty dx\, x^p e^{-x^2} = \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right) \quad ; \quad p > -1$$

$$\therefore \int_0^\infty dx\, e^{-x^2} = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty dx\, x^2 e^{-x^2} = \frac{1}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\cdot\frac{1}{2} = \frac{\sqrt{\pi}}{4}$$

7. Stirling's Formula

$$\ln\Gamma(z) \simeq \left(z - \tfrac{1}{2}\right)\ln z - z + \tfrac{1}{2}\ln 2\pi + \mathcal{O}\left(\tfrac{1}{z}\right) \quad \text{for } z \to \infty,\ |\arg(z)| < \pi$$