

Thermodynamic Potential for Superfluid ^3He in Aerogel

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Abstract We present a free energy functional for superfluid ^3He in the presence of homogeneously distributed impurity disorder which extends the Ginzburg-Landau free energy functional to all temperatures. We use the new free energy functional to calculate the thermodynamic potential, entropy, heat capacity and density of states for the B-phase of superfluid ^3He in homogeneous, isotropic aerogel.

Keywords Superfluid ^3He · Aerogel · Gapless superfluid · Thermodynamics

1 Introduction

Liquid ^3He infused into high porosity silica aerogel provides us with a model physical system for studying the effects of quenched disorder on an Fermi liquid with unconventional pairing [1]. Indeed ^3He -aerogel provides us with a system where the effects of disorder can be explored over the entire range from weak to strong disorder without modifying the basic interactions of the Fermi liquid.

Theoretical analysis by Thuneberg et al. [2] showed that the effects of scattering by a homogeneous, isotropic medium enhanced the stability of the Balian-Werthamer (BW) phase relative to the anisotropic ABM or Planar phase. These authors also calculated the reduction in the transition temperature, T_c , the order parameter, $\hat{\Delta}(\hat{\mathbf{p}})$, and condensation energy, $\Delta\Omega(p, T)$, in the Ginzburg-Landau (GL) limit. Experimentally, the equilibrium phase of ^3He -aerogel is consistent with the Balian-Werthamer

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(BW) state, modified by de-pairing, over the entire phase diagram, except perhaps in a narrow region of temperatures near T_c at high pressures [3–5]. Measurements of the heat capacity of ^3He -aerogel at intermediate and high pressures provide quantitative measurements of de-pairing, including the reduction in $\Delta C/\gamma_N T_{c0}$ as well as indirect evidence of gapless excitations with a finite density of states, $N(0)$, at the Fermi energy [6].

We report results based on a free energy functional which extends the GL theory for superfluid ^3He in the presence of homogeneously distributed impurity disorder to all temperatures. This functional is obtained by a reduction of the Luttinger-Ward functional to leading order in the expansion parameters of Fermi-liquid theory. Results for the thermodynamic potential, entropy, heat capacity and density of states for the B-phase of superfluid ^3He -aerogel in the weak-coupling limit are reported. For impurity scattering in the unitary limit the B-phase exhibits gapless behavior over the entire phase diagram for high-porosity aerogels with an elastic mean-free path $\ell_{el} = 1500 \text{ \AA}$.

2 Free Energy Functional

Serene and Rainer (SR) obtained a functional of the low-energy, quasiclassical propagator, $\widehat{g}(\hat{\mathbf{p}}, \varepsilon_n)$, and self-energy, $\widehat{\sigma}(\hat{\mathbf{p}}, \varepsilon_n)$, for a superfluid Fermi liquid starting from the Luttinger-Ward-DeDominicis-Martin (LWDM) functional [7, 8] of the exact propagator and self energy and subtracting the functional evaluated at the stationarity condition corresponding to the normal Fermi liquid [9]. This subtraction confines the functional to the low-energy states near the Fermi surface.¹ Note that $\widehat{\sigma}$, represents corrections to the leading order normal-state self-energy. Similarly, the subtracted Φ -functional can be expressed as a functional of the low-energy propagator and effective interactions represented by block vertices that sum to all orders the bare interaction and higher order processes mediated by the high-energy propagator. So long as the block vertices do not introduce a new low-energy scale, then the contributions to the subtracted Φ -functional can be classified by their order in one or more small parameters of Fermi-liquid theory. The SR functional has the form,

$$\Delta\Omega[\widehat{g}, \widehat{\sigma}] = -\frac{1}{2} Sp' \{ \widehat{\sigma} \widehat{g} \} + \Delta\Omega_{ln} + \Phi[\widehat{g}], \quad (1)$$

$$\Delta\Omega_{ln} = -\frac{1}{2} Sp' \left\{ \int d\xi_{\mathbf{p}} [\ln_{\otimes}(-i\varepsilon_n \widehat{\tau}_3 + \xi_{\mathbf{p}} + \widehat{\sigma}) - \ln_{\otimes}(-i\varepsilon_n \widehat{\tau}_3 + \xi_{\mathbf{p}})] \right\}, \quad (2)$$

where $Sp' \{ \dots \} \equiv 2N_f \int d^3R \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} \pi T \sum_{\varepsilon_n} \text{Tr}_4 \{ \dots \}$ and $\text{Tr}_4 \{ \dots \}$ is the trace in Nambu space. The SR functional is general enough to describe inhomogeneous equilibrium states of a superfluid Fermi liquid which vary slowly on the atomic

¹We follow the notation in Ref. [9], except that we denote 4×4 matrices in Nambu space by a widehat, e.g. \widehat{g} , while narrow hats refer to unit vectors, e.g. $\hat{\mathbf{p}}$, is the unit vector in the direction of the Fermi momentum, $\mathbf{p}_f = p_f \hat{\mathbf{p}}$.

scale. In such cases the convolution product attached to the ln-functional is defined as $\widehat{a} \otimes \widehat{b}(\mathbf{p}, \mathbf{R}) = \exp\{\frac{i}{2}(\nabla_{\mathbf{R}}^a \cdot \nabla_{\mathbf{p}}^b - \nabla_{\mathbf{p}}^a \cdot \nabla_{\mathbf{R}}^b)\} \widehat{a}(\mathbf{p}, \mathbf{R}) \widehat{b}(\mathbf{p}, \mathbf{R})$. For example $\xi_{\mathbf{p}} \otimes \widehat{a} = (\xi_{\mathbf{p}} - \frac{i}{2} \mathbf{v}_f \cdot \nabla_{\mathbf{R}}) \widehat{a}(\mathbf{p}, \mathbf{R})$. The SR functional is stationary with respect to variations of \widehat{g} and $\widehat{\sigma}$. The latter condition generates,

$$\widehat{g}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) = \int d\xi_{\mathbf{p}} (i\widehat{\tau}_3 \varepsilon_n - \xi_{\mathbf{p}} - \widehat{\sigma})_{\otimes}^{-1}, \tag{3}$$

which can be transformed à la Eilenberger into the quasiclassical transport equation [10], $[i\widehat{\tau}_3 \varepsilon_n - \widehat{\sigma}, \widehat{g}] + i\mathbf{v}_f \cdot \nabla_{\mathbf{R}} \widehat{g} = 0$, and normalization condition, $\widehat{g}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n)^2 = -\pi^2 \widehat{1}$. Stationarity of the SR functional with respect to variations of the propagator, $\widehat{g} \rightarrow \widehat{g} + \delta\widehat{g}$, identifies the self energy with the skeleton expansion obtained from the Φ -functional,

$$\delta\Phi[\widehat{g}] \equiv \Phi[\widehat{g} + \delta\widehat{g}] - \Phi[\widehat{g}] = \frac{1}{2} Sp' \{ \widehat{\sigma} \delta\widehat{g} \}. \tag{4}$$

To leading order in the small expansion parameter the Φ -functional is given by

$$\Phi_{\text{ff}} = N_f \int d^3R \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} v(\hat{\mathbf{p}}, \hat{\mathbf{p}}') T \sum'_{\varepsilon_n} \mathbf{f}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) \cdot T \sum'_{\varepsilon'_n} \bar{\mathbf{f}}(\hat{\mathbf{p}}', \mathbf{R}; \varepsilon'_n), \tag{5}$$

where $v(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ is the odd-parity, spin-triplet pairing interaction, while $\mathbf{f}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n)$ and $\bar{\mathbf{f}}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n)$ are the spin-triplet components of the off-diagonal “anomalous” components of the propagator. This functional generates the mean-field pairing self energy,

$$\Delta(\hat{\mathbf{p}}, \mathbf{R}) = \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} v(\hat{\mathbf{p}}, \hat{\mathbf{p}}') T \sum'_{\varepsilon'_n} \mathbf{f}(\hat{\mathbf{p}}', \mathbf{R}; \varepsilon'_n). \tag{6}$$

The Matsubara sums include a cutoff, Ω_c , that restricts the pairing interaction to the low-energy band near the Fermi level. At this same order the diagonal mean-fields, i.e. the Landau molecular fields, also contribute. These terms are important for states perturbed by external fields or for inhomogeneous phases, but vanish for the homogeneous equilibrium state. For the applications discussed below the Landau molecular field self energy is omitted.

In order to obtain a generalization of the GL functional we invert the self-consistency condition for the anomalous propagator to reduce the SR functional to a functional of the order parameter, Δ . Thus, Φ_{ff} can be evaluated to give,

$$\Phi_{\text{ff}} = N_f \int d^3R \frac{1}{v_1} \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} |\Delta(\hat{\mathbf{p}}, \mathbf{R})|^2, \tag{7}$$

for pure p -wave pairing. Similarly, in (1), $-\frac{1}{2} Sp' \{ \widehat{\sigma} \widehat{g} \} = -2\Phi_{\text{ff}}$.

In order to include the interaction of liquid ^3He with the structure of silica aerogel we explicitly include a one-body interaction, $\widehat{V} = \widehat{V}\widehat{1}$, representing the interaction of quasiparticles with the static impurity potential, $V(\mathbf{p}, \mathbf{p}'; \{\mathbf{R}_i\})$. The impurity potential enters via the ln-functional, and the stationarity condition with respect to $\widehat{\sigma}$ generates (3), but now with the impurity potential as an addition to the self energy $\widehat{\sigma}$. The

impurity potential depends on the static configuration of the impurities, $\{\mathbf{R}_i\}$, which are treated as random variables governed by a probability distribution, $P(\{\mathbf{R}_i\})$. The impurity-averaged propagator is defined by averaging the propagator for a specific configuration of impurities over the probability distribution for a particular configuration,

$$\widehat{g}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) = \int d\xi_{\mathbf{p}} \prod_{i=1}^{N_s} \int d^3 R_i P(\{\mathbf{R}_i\}) (i\varepsilon_n \widehat{\tau}_3 - \xi_{\mathbf{p}} - \widehat{\Delta} - \widehat{V}(\{\mathbf{R}_i\}))_{\otimes}^{-1}. \tag{8}$$

The impurity average generates an additional self energy term describing the scattering of quasiparticles and correlated pairs by the random potential. The derivation of the transport equation carries through for the ensemble averaged propagator, which now takes the form

$$[i\widehat{\tau}_3 \varepsilon_n - \widehat{\Delta} - \widehat{\sigma}_{\text{imp}}, \widehat{g}] + i\mathbf{v}_f \cdot \nabla_{\mathbf{R}} \widehat{g} = 0. \tag{9}$$

For uncorrelated impurities the impurity self energy, to leading order in $1/\tau E_f$, is determined by mean density of impurities and the self-consistent t-matrix for impurity scattering,

$$\widehat{\sigma}_{\text{imp}}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) = n_{\text{imp}} \widehat{t}(\hat{\mathbf{p}}, \hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) \tag{10}$$

$$\widehat{t}(\hat{\mathbf{p}}, \hat{\mathbf{p}}', \mathbf{R}; \varepsilon_n) = \widehat{u}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') + N_f \int \frac{d\Omega_{\hat{\mathbf{p}}''}}{4\pi} \widehat{u}(\hat{\mathbf{p}}, \hat{\mathbf{p}}'') \widehat{g}(\hat{\mathbf{p}}'', \mathbf{R}; \varepsilon_n) \widehat{t}(\hat{\mathbf{p}}'', \hat{\mathbf{p}}', \mathbf{R}; \varepsilon_n). \tag{11}$$

For the results presented here we consider isotropic impurities with dominant scattering in the s-wave channel, i.e. $\widehat{u} = u_0 \widehat{1}$. In this case the t-matrix reduces to,

$$\widehat{t}(\mathbf{R}; \varepsilon_n) = u_0 (\widehat{1} + N_f \langle \widehat{g}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) \rangle_{\hat{\mathbf{p}}} \widehat{t}(\mathbf{R}; \varepsilon_n)), \tag{12}$$

where $\langle \widehat{g}(\hat{\mathbf{p}}, \mathbf{R}; \varepsilon_n) \rangle_{\hat{\mathbf{p}}}$ is the average of the propagator over the Fermi surface. For normal ^3He the propagator reduces to $\widehat{g}_N = -i\pi \widehat{\tau}_3 \text{sgn}(\varepsilon_n)$, and the normal-state t-matrix is parameterized by the s-wave scattering phase shift, $\delta_0 = \tan^{-1}(\pi N_f u_0)$, and reduces to $\widehat{t}_N(\varepsilon_n) = \frac{1}{\pi N_f} \sin \delta_0 e^{-i s_\varepsilon \delta_0 \widehat{\tau}_3}$, where $s_\varepsilon = \text{sgn}(\varepsilon_n)$. In this minimal model for aerogel scattering, the mean density of impurities $n_s = 1/\sigma \ell$, and the scattering rate, $1/\tau$, for normal-state quasiparticles are fixed by the mean free path, $\ell = v_f \tau$, and scattering cross-section, $\sigma = 4\pi \lambda_f^2 \bar{\sigma}_0$ where $\lambda_f = \hbar/p_f$ is the Fermi wavelength and $\bar{\sigma}_0 = \sin^2 \delta_0$ is the dimensionless cross-section [12].

In order to average the free energy functional in (1) over the impurity ensemble we must average the ln-functional in (2). This is facilitated by the integral representation for the ln-functional,

$$\begin{aligned} & -\text{Tr}_4 \{ \ln_{\otimes} (-i\varepsilon_n \widehat{\tau}_3 + \xi_{\mathbf{p}} + \widehat{\Delta}(\mathbf{p}, \mathbf{R}) + \widehat{V}(\{\mathbf{R}_i\})) \} \\ & = \text{Tr}_4 \left\{ i\widehat{\tau}_3 \int_{\varepsilon_n}^{\infty} d\lambda (i\lambda \widehat{\tau}_3 - \xi_{\mathbf{p}} - \widehat{\Delta}(\hat{\mathbf{p}}, \mathbf{R}) - \widehat{V}(\{\mathbf{R}_i\}))_{\otimes}^{-1} \right\}. \end{aligned} \tag{13}$$

For the subtracted functional we can now carry out the ensemble average, and interchange the $\xi_{\mathbf{p}}$ integration with the auxiliary integration over the Matsubara energy to obtain the ensemble averaged free energy functional,

$$\Delta\overline{\Omega}[\widehat{\Delta}] = N_f \int d^3R \int \frac{d\Omega_{\widehat{\mathbf{p}}}}{4\pi} \left\{ -\frac{1}{v_1} |\mathbf{\Delta}(\widehat{\mathbf{p}}, \mathbf{R})|^2 + \frac{1}{2} T \sum'_{\varepsilon_n} \int_{\varepsilon_n}^{\infty} d\lambda \text{Tr}_4 \{ i\widehat{\tau}_3 [\widehat{\mathbf{g}}(\widehat{\mathbf{p}}, \mathbf{R}; \lambda) - \widehat{\mathbf{g}}_N(\lambda)] \} \right\}, \tag{14}$$

where $\widehat{\mathbf{g}}_N(\lambda) = -i\pi\widehat{\tau}_3 \text{sgn}(\lambda)$ and $\widehat{\mathbf{g}}(\widehat{\mathbf{p}}, \mathbf{R}; \lambda)$ is obtained from the solution to the (9), (10) and (12) for fixed $\widehat{\Delta}(\widehat{\mathbf{p}}, \mathbf{R})$ and auxiliary Matsubara energy, $i\lambda$. This free energy functional can be used to evaluate the thermodynamic potential for a broad range of equilibrium states of unconventional superconductors in the presence of impurity disorder; it also represents an extension of the Eilenberger’s free energy functional for conventional superconducting alloys in the Born impurity scattering limit [13]. As an application we calculate the thermodynamic potential for the BW state in the presence of homogeneous, isotropic impurity scattering as a model for the B-phase of ^3He in silica aerogel.

For spatially uniform states the solution to the transport equation and normalization condition for the propagator can be expressed as

$$\widehat{\mathbf{g}} = -\pi \frac{i\widetilde{\varepsilon}_n \widehat{\tau}_3 - \widetilde{\Delta}(\widehat{\mathbf{p}}; \varepsilon_n)}{\sqrt{\widetilde{\varepsilon}_n^2 + |\widetilde{\Delta}(\widehat{\mathbf{p}}; \varepsilon_n)|^2}}, \tag{15}$$

where $\widetilde{\varepsilon}_n = \varepsilon_n + i n_{\text{imp}} t_3(\varepsilon_n)$ and $\widetilde{\Delta} = \mathbf{\Delta}(\widehat{\mathbf{p}}) + \mathbf{\Delta}_{\text{imp}}(\varepsilon_n)$ are the Matsubara energy and order parameter renormalized by the impurity self-energy. However, $\mathbf{\Delta}_{\text{imp}}(\varepsilon_n)$ vanishes for pure s-wave scattering because the order parameter vanishes when averaged over the Fermi surface, i.e. $\langle \mathbf{\Delta}(\widehat{\mathbf{p}}) \rangle_{\widehat{\mathbf{p}}} = 0$ [11]. Thus, we obtain from the $\widehat{\tau}_3$ -component of the t-matrix,

$$\widetilde{\varepsilon}_n = \varepsilon_n + \frac{1}{2\tau} \left\langle \frac{\widetilde{\varepsilon}_n \sqrt{\widetilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\widehat{\mathbf{p}})|^2}}{\widetilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\widehat{\mathbf{p}})|^2 (1 - \bar{\sigma}_0)} \right\rangle_{\widehat{\mathbf{p}}}. \tag{16}$$

The self-consistency equation for the order parameter is obtained by evaluating (6) with the solution for $\mathbf{f}(\widehat{\mathbf{p}}; \varepsilon_n)$ obtained from (15). For unitary spin-triplet states ($\mathbf{\Delta} \times \mathbf{\Delta}^* = 0$) the gap equation becomes,

$$\mathbf{\Delta}(\widehat{\mathbf{p}}) = - \int \frac{d\Omega_{\widehat{\mathbf{p}}'}}{4\pi} v(\widehat{\mathbf{p}}, \widehat{\mathbf{p}}') \pi T \sum'_{\varepsilon_n} \frac{\mathbf{\Delta}(\widehat{\mathbf{p}}')}{\sqrt{\widetilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\widehat{\mathbf{p}}')|^2}}. \tag{17}$$

For pure ^3He the linearized gap equation (LGE) with $\mathbf{\Delta} \rightarrow 0$ and $1/2\tau \rightarrow 0$ determines the bulk transition temperature, T_{c0} , in terms of the p-wave pairing interaction, v_1 , and the bandwidth of low-energy states, Ω_c , i.e. $\mathbf{K}(T_{c0}) = -1/v_1$, with $\mathbf{K}(T) = 2\pi T \sum'_{\varepsilon_n \geq 0} (1/\varepsilon_n) \simeq \ln(1.13\Omega_c/T)$. The LGE also regulates the Matsubara sum and removes the pairing interaction and cutoff in favor of the pure ^3He transition

temperature, T_{c0} .

$$\ln\left(\frac{T}{T_{c0}}\right)\Delta(\hat{\mathbf{p}}) = 2\pi T \sum_{n \geq 0} \left(\frac{1}{\sqrt{\tilde{\varepsilon}_n^2 + |\Delta(\hat{\mathbf{p}})|^2}} - \frac{1}{\varepsilon_n} \right) \Delta(\hat{\mathbf{p}}). \tag{18}$$

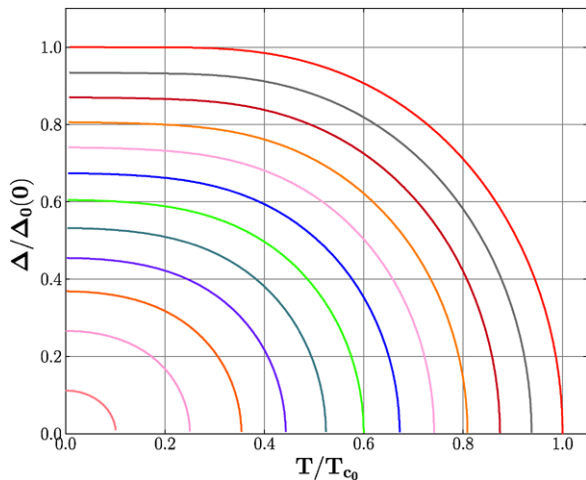
The renormalization of the Matsubara energy encodes the effects of de-pairing by the scattering of pair-correlated quasiparticles off the distribution of impurities. This is evident in suppression of the superfluid transition and the order parameter. The suppression of the superconducting transition is given by the solution of (18) for $|\Delta| \rightarrow 0^+$. In this limit, $\tilde{\varepsilon}_n \rightarrow \varepsilon_n + \frac{1}{2\tau} \text{sgn}(\varepsilon_n)$ and $T \rightarrow T_c$,

$$\ln\left(\frac{T_c}{T_{c0}}\right) = \sum_{n \geq 0} \left(\frac{1}{n + \frac{1}{2} + \frac{1}{2} \frac{1}{2\pi\tau T_c}} - \frac{1}{n + \frac{1}{2}} \right), \tag{19}$$

and we obtain the Abrikosov-Gorkov formula [2]. Note that the pair-breaking parameter, $\alpha = 1/2\pi\tau T_{c0} = \xi_0(p)/\ell$, where $\xi_0 = \hbar v_f / 2\pi k_B T_{c0}$ is the pure ^3He coherence length and $\ell = v_f \tau$ is identified with the transport mean free path. The full range from weak to strong pair breaking can be explored by varying the pressure and aerogel density. The critical point in the phase diagram where $T_c = 0$ is given by $\alpha_c \simeq 0.281$. For 98% aerogel with a mean free path of $\ell \simeq 1400 \text{ \AA}$ this gives a critical pressure of $p_c \simeq 6 \text{ bar}$, below which the superfluid transition is completely suppressed [14]. For superfluidity to survive down to $p = 0 \text{ bar}$ the mfp must exceed $\approx 3000 \text{ \AA}$, while for $\ell \lesssim 650 \text{ \AA}$ superfluidity is completely suppressed for all pressures.

For the B-phase, $|\Delta(\hat{\mathbf{p}})|^2 = |\Delta|^2$ is isotropic, the angular integration over the Fermi surface is trivial and we can solve the gap equation numerically to obtain the magnitude of the order parameter. Figure 1 shows the de-pairing effect on both the transition temperature and the magnitude of the order parameter over the full temperature range as a function of α for unitary scattering.

Fig. 1 (Color online) Suppression of the B-phase order parameter in the unitary limit as a function of $\alpha = 1/2\pi\tau T_{c0}$. $\Delta(T)$ is normalized by $\Delta_0(0) = 1.78k_B T_{c0}$ for pure $^3\text{He-B}$ in the weak-coupling limit (red curve). The curves for $0 < \alpha \leq 0.275$ are shown for α in steps of 0.025



The free energy functional for spatially uniform phases of ³He-aerogel is obtained by carrying out the integration in (14) over the impurity renormalized propagator in (15). For unitary spin-triplet states and impurity scattering in the unitary scattering limit we obtain,

$$\Delta\bar{\Omega}[\mathbf{\Delta}, \mathbf{\Delta}^*] = N_f V \left\langle -\frac{1}{v_1} |\mathbf{\Delta}(\hat{\mathbf{p}})|^2 + 2\pi T \sum_{\varepsilon_n} \left[\left(|\tilde{\varepsilon}_n| - \sqrt{\tilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\hat{\mathbf{p}})|^2} \right) \times \left(1 + \frac{1}{2\tau |\tilde{\varepsilon}_n|} \right) + \frac{1}{2\tau} \ln \left(\frac{\sqrt{\tilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\hat{\mathbf{p}})|^2}}{|\tilde{\varepsilon}_n|} \right) \right] \right\rangle_{\hat{\mathbf{p}}}, \tag{20}$$

$$\tilde{\varepsilon}_n = \varepsilon_n + \left\langle \frac{1}{2\tau} \frac{\sqrt{\tilde{\varepsilon}_n^2 + |\mathbf{\Delta}(\hat{\mathbf{p}})|^2}}{\tilde{\varepsilon}_n} \right\rangle_{\hat{\mathbf{p}}}. \tag{21}$$

This functional reduces to the weak-coupling functional for pure ³He (Eq. 5.16 of Ref. [9]) for $\tau \rightarrow \infty$. One can also verify that the stationarity condition for the ensemble averaged functional, $\delta\Delta\bar{\Omega}/\delta\mathbf{\Delta}(\hat{\mathbf{p}})^* = 0$, generates the impurity renormalized gap equation, i.e. (17).

The thermodynamic potential, entropy and heat capacity of BW phase with homogeneous, isotropic disorder are obtained by evaluating the ensemble averaged free energy functional with the self-consistently determined solutions of (18), (20) and (21).

3 Thermodynamic Functions

Figure 2 shows the thermodynamic potential (left panel) and entropy (right panel) for the dirty B-phase for values of the pair breaking parameter spanning the range from weak to strong pair breaking. Note that we plot the full thermodynamic po-

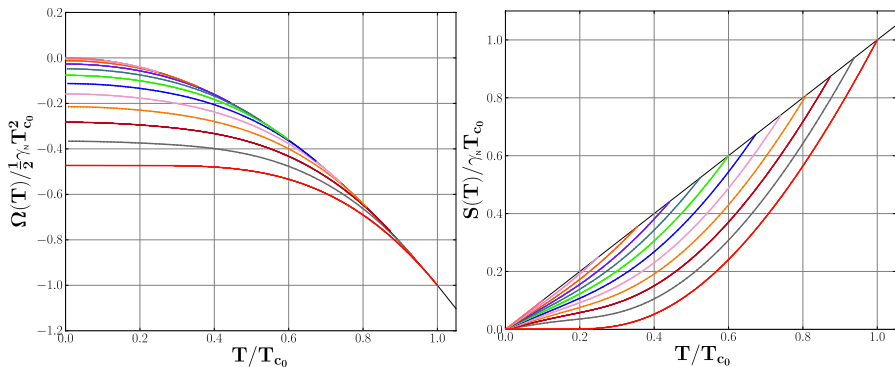


Fig. 2 (Color online) Thermodynamic potential (left panel) and entropy (right panel) for the B-phase of ³He-aerogel in the unitary limit as a function of T/T_{c0} for pair breaking ranging from $0 \leq \alpha \leq 0.275$ in steps of 0.025. Results for pure ³He-B are shown as red curves

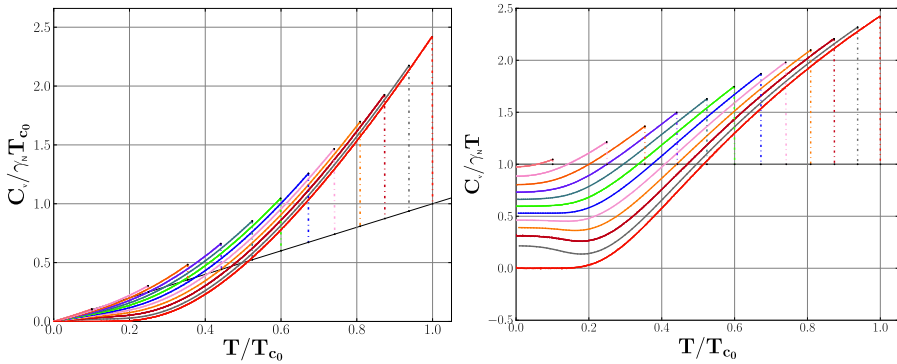


Fig. 3 (Color online) Specific heat capacity normalized by $\gamma_N T_{c_0}$ (left panel) and by $\gamma_N T$ (right panel) for the B-phase of ^3He -aerogel in the unitary limit as a function of T/T_{c_0} for pair breaking ranging from $0 \leq \alpha \leq 0.275$ in steps of 0.025. Results for pure ^3He -B are shown as red curves

tential, $\Omega_S(T) = \Omega_N(T) + \Delta\bar{\Omega}(T)$, which includes the normal-state contribution $\Omega_N(T) = -\frac{1}{2}\gamma_N T^2$, and we normalized the potential in units of $\frac{1}{2}\gamma_N T_{c_0}^2$, where $\gamma_N = \frac{2}{3}\pi^2 N_f$ is the Sommerfeld coefficient for the normal state. The entropy was computed by numerically differentiating the thermodynamic potential, $S(T) = -\partial\Omega/\partial T|_V$. For pure ^3He -B the entropy vanishes exponentially for $T \ll T_{c_0}$, but for the disordered B-phase in the unitary limit the entropy vanishes linearly for sufficiently low temperatures, $S(T) \rightarrow \gamma_S T$, where γ_S is the Sommerfeld coefficient for the gapless B-phase. The gapless regime is achieved for $T \ll \gamma_{\text{imp}}$, where $\gamma_{\text{imp}} \simeq \sqrt{\Delta(0)/2\tau}$ is the bandwidth of the impurity-induced excitation spectrum near the Fermi level.

The heat capacity provides both a quantitative measure of the order that develops at T_c , as well as the spectrum of gapless states near the Fermi level. Figure 3 (left panel) shows results for the specific heat, $C_V = T\partial S/\partial T|_V$ over the full temperature range. The reduction of the heat capacity jump at T_c computed from the thermodynamic potential agrees with the value calculated from the GL theory of Ref. [2] to three significant figures for all values of the pair breaking parameter. The right panel is the same data for the heat capacity plotted as $C_V/\gamma_N T$ vs. T/T_{c_0} . For weak to modest pair-breaking, $\alpha = 1/2\pi\tau T_{c_0} < 0.1$, corresponding to $\ell_{\text{el}} > 1600 \text{ \AA}$ at $p = 33 \text{ bar}$, C_V/T is a non-monotonic function of T , however the limiting low-temperature heat capacity is linear in temperature, $C_V \rightarrow \gamma_S T$, where γ_S is the Sommerfeld coefficient for temperatures below the impurity bandwidth, $T \lesssim \gamma_{\text{imp}}$. The non-monotonic behavior of C_V/T for $\alpha \lesssim 0.1$ results from the gap between the impurity band and the continuum states with $\varepsilon \geq |\Delta|$ (see Fig. 4). The limiting value provides the Sommerfeld coefficient for the gapless B-phase, $\gamma_S = \gamma_N \lim_{T \rightarrow 0} C_V/\gamma_N T$. One can infer from the calculated Sommerfeld coefficient for the gapless B-phase the density of states at the Fermi level, i.e. $\gamma_S = \frac{2}{3}\pi^2 N(0)$.

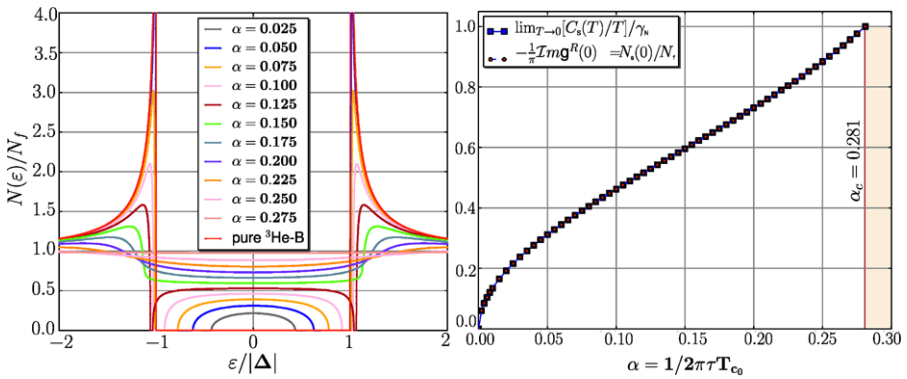


Fig. 4 (Color online) Density of states as a function of excitation energy from weak to strong pair breaking (left). Comparison of the normalized density of states at the Fermi level (right panel), $N(0)/N_f$ obtained by direct calculation from (24), and the Sommerfeld ratio, γ_S/γ_N computed from the limit, $\lim_{T \rightarrow 0} C_S/T$

4 Density of States

On the other hand, the density of states can be calculated directly by analytic continuation of the Matsubara propagator to real energies,

$$g(i\varepsilon_n) \xrightarrow{i\varepsilon_n \rightarrow \varepsilon + i0^+} g^R(\varepsilon) = -\pi \frac{\tilde{\varepsilon}^R(\varepsilon)}{\sqrt{|\Delta|^2 - (\tilde{\varepsilon}^R(\varepsilon))^2}}, \tag{22}$$

$$\tilde{\varepsilon}^R = \varepsilon + i0^+ - \frac{1}{2\tau} \frac{\sqrt{|\Delta|^2 - (\tilde{\varepsilon}^R(\varepsilon))^2}}{\tilde{\varepsilon}^R}. \tag{23}$$

$$N(\varepsilon) = N_f \left(-\frac{1}{\pi} \text{Im} g^R(\varepsilon) \right), \tag{24}$$

where $N(\varepsilon)$ is the density of states (DOS) at excitation energy ε measured relative to the Fermi level. Figure 4 shows the results for the DOS (left panel) from weak to strong pair breaking calculated in the unitary limit for $T \rightarrow 0$. Note that the excitation energy for each value of pairbreaking is normalized by the appropriate value of the order parameter, $\Delta(0)$. The divergence of $N(\varepsilon)$ at $\varepsilon = \pm|\Delta|$ is suppressed by impurity scattering, and gapless excitations are created near the Fermi level for any $\alpha \neq 0$ in the unitary scattering limit.

The normalized density of states at $\varepsilon = 0$ can be compared with that inferred from the calculated from the Sommerfeld coefficient. This provides a strong test of our calculations for the thermodynamic potential, entropy and heat capacity, as well as the validity of the ensemble averaged free energy functional obtained in (20). As Fig. 4 shows the results for $N(0)/N_f$ and γ_S/γ_N (right panel) are in perfect agreement. Note that $N(0)/N_f$ and γ_S/γ_N vanish for $\alpha = 0$ and both approach 1 continuously for $\alpha \rightarrow \alpha_c \approx 0.281$.

In summary we have obtained an ensemble averaged free energy functional for the superfluid ^3He in the presence of homogeneous, isotropic impurity disorder. This

functional is based on a quasiclassical reduction of the Luttinger-Ward functional. Results for the thermodynamic potential, entropy and heat capacity were obtained by numerical evaluation of the stationarity conditions. The predicted Sommerfeld ratio based on the new functional agrees exactly with a direct calculation of the DOS from a solution of the quasiclassical transport equation. A quantitative comparison between theory and experimental measurements of the heat capacity of superfluid $^3\text{He-B}$ in aerogel [6] made at intermediate and high pressures will be discussed in a separate report.

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