Magnetic vortices in a rotating $^3P_2$ neutron superfluid

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$^3P_2$ vortices in neutron-star matter are structurally different from vortices in a $^1S_0$ superfluid. The most important difference is that $^3P_2$ vortices have a spontaneous magnetization in the core region directed along the vortex line. We estimate the velocity relaxation time for electron scattering from these vortices and compare our results with those of Feibelman and the observed relaxation times for the Crab and Vela pulsars.

INTRODUCTION

The observation of macroscopic time scales for glitch relaxation has provided convincing evidence that neutron stars contain superfluids.[1] When combined with calculations of the structure and properties of neutron stars, pulsar timing data may serve to determine the type and distribution of neutron superfluids as well as their dominant interactions with the rest of the star.

Theoretical calculations suggest that in the inner crust (3×10^{11} g/cm$^3 \lesssim \rho \lesssim 2\times10^{14}$ g/cm^3) the neutrons condense into a BCS-type superfluid in which the neutron pairs form s-wave ($^1S_0$) bound states. At higher densities in the interior, the neutron pairs are expected to condense in a $^3P_2$ state.[2] These superfluids must be threaded by an array of quantized vortices in order to participate in the rotation of the star. The motion of these vortices during the deceleration of the star and their response to the discontinuous speedup of the crust and conducting fluid during a glitch are determined by frictional and pinning forces acting on the vortex lines.

Theories of neutron superfluid spin down (or spin up) have concentrated on the dynamics of the crust superfluid or have assumed that $^3P_2$ vortices are similar to conventional s-wave vortices.[3,4] In this article we discuss the structure of $^3P_2$ vortices in relation to neutron-star rotational dynamics. We note that a smooth interpolation between Richardson's vortex solutions to the Ginzburg-Landau equations[5] near the vortex core and the asymptotic vortex solution of Muzikar *et al.*[6] implies that $^3P_2$ vortices have a small spontaneous magnetization in the region near the core. This magnetization differentiates $^3P_2$ vortices from their s-wave counterparts and may be relevant to velocity relaxation of the neutron-star interior. In the latter part of this article we estimate the velocity relaxation time due to electron scattering from magnetic $^3P_2$ vortices. We compare this time with the relaxation time due to electron quasiparticle scattering calculated by Feibelman[4] and with the observed relaxation times in the Crab and Vela pulsars.

In an s-wave Fermi superfluid axial-symmetric 2π-vortex lines are described by a complex scalar order parameter.

$$\Delta(\vec{R})=\Delta(\rho)e^{i\phi}\ , \tag{1}$$

where ρ, ϕ, z are cylindrical coordinates measured from the center of a vortex line along $\hat{z}$. The phase changes by 2π in one circuit around the line, and the corresponding superfluid velocity $\vec{v}_s=(\hbar/2m\rho)\hat{\phi}$ yields a single quantum of circulation, $\kappa\equiv h/2m$. The structure of s-wave vortices results from the competition between two energies: the condensation energy determined by $\Delta(\rho)$ and the kinetic energy of the supercurrents determined by $\vec{v}_s$. At large distances from the center of the line the kinetic energy of the supercurrents is small compared with the condensation energy; consequently, we have $\Delta(\rho\to\infty)=\Delta(T)$, where $\Delta(T)$ is the equilibrium energy gap. The distance where the kinetic energy becomes comparable to the condensation energy determines the radius of the vortex core, which is of the order of the temperature-dependent coherence length ξ_T. In order to reduce the kinetic energy inside the core, $0\leq\rho\lesssim\xi_T$, the amplitude of the order parameter decreases as $\Delta(\rho)/\Delta(T) = \text{const} \times \rho/\xi_T$. The vanishing of $\Delta(\rho)$ for $\rho \to 0$ is the only mechanism available in an s-wave superfluid for reducing the kinetic energy associated with the divergent velocity field.

$^3P_2$ VORTEX STRUCTURE

Compared to their s-wave counterparts, the structure of $^3P_2$ vortices is complicated by the existence of other order-parameter states, besides the equilibrium state, which are stationary points of the uniform free-energy functional.[7] This structure is conveniently described by a Cartesian tensor order parameter $A_{\mu\nu}(\vec{R})$ which is traceless and symmetric, and related to the 2×2 gap matrix

$$\Delta_{\alpha\beta}(\hat{k};\vec{R}) = \sum_{\mu,\nu=1}^{3} (i\sigma^\mu\sigma^2)_{\alpha\beta} A_{\mu\nu}(\vec{R})\,\hat{k}_\nu \,, \tag{2}$$

where $\vec{k} = k_F\hat{k}$ and $\vec{R}$ are the relative momenta and center-of-mass coordinate of a pair.

For distances far from the core the condensation energy is large compared with kinetic energy so that the asymptotic $^3P_2$ vortex structure is described by the local equilibrium order parameter[6]

$$A_{\mu\nu}(\vec{R}) = \frac{3}{\sqrt{2}}\Delta(T)e^{i\phi}(\hat{u}_\mu\hat{u}_\nu - \tfrac{1}{3}\delta_{\mu\nu}) \,, \tag{3}$$

where $\Delta(T)$ is the angle-averaged energy gap,

$$\Delta(T)^2 = \tfrac{1}{2}\int \frac{d\Omega}{4\pi}\,\mathrm{tr}[\hat{\Delta}(\hat{k})\hat{\Delta}^\dagger(\hat{k})] \,,$$

$\hat{u}$ is the local axis along which the pairs have $M_J = 0$. Equation (3) is analogous to the asymptotic limit of Eq. (1) for the s-wave vortex, in which the (scalar) order parameter is given by the equilibrium gap multiplied by a phase factor describing the supercurrents. However, $^3P_2$ vortices have additional structure because the order parameter is a tensor. In the asymptotic limit this structure is determined by minimizing the sum of the kinetic and bending energies associated with spatial variations of the phase and $\hat{u}$. For a uniform $\hat{u}$ the bending energy is zero; the total gradient energy[8]

$$\Delta\Omega_g = k_1(\partial_\mu A_{\alpha\nu})(\partial_\mu A^*_{\alpha\nu}) + k_2(\partial_\mu A_{\alpha\nu})(\partial_\nu A^*_{\alpha\mu}) + k_3(\partial_\mu A_{\alpha\mu})(\partial_\nu A^*_{\alpha\nu}) \tag{4}$$

is purely kinetic and can be written as

$$\begin{aligned}
\Delta\Omega_{\text{kinetic}} &= \tfrac{1}{2}(\rho_s)_{\alpha\beta}(v_s)_\alpha(v_s)_\beta \,,\\
(\rho_s)_{\alpha\beta} &= \rho_1\delta_{\alpha\beta} + \rho_2\hat{u}_\alpha\hat{u}_\beta \,,\\
\rho_1 &= 6\left[\frac{2m}{\hbar}\right]^2 \Delta^2(k_1 + \tfrac{1}{6}k_{23}) \,,\\
\rho_2 &= 3\left[\frac{2m}{\hbar}\right]^2 \Delta^2 k_{23} \,.
\end{aligned} \tag{5}$$

In BCS theory $\rho_{1,2} > 0$ so that $\hat{u}\,||\,\hat{z}$ minimizes the kinetic energy, and also the total gradient energy since there is no competition between kinetic and bending energy.

The structure of $^3P_2$ vortices in the region near the core is further complicated because there are two length scales for spatial variations of the order parameter.[6] Like its s-wave counterpart, the core radius is of the order of the temperature-dependent coherence length ξ_T and corresponds to the distance at which the gradient energy is comparable to the condensation energy. The additional length scale occurs because the unitary states, described by a real (up to an overall phase factor) order parameter

$$A_{\mu\nu} = N\Delta(T)e^{i\chi}[\hat{u}_\mu\hat{u}_\nu + r\,\hat{v}_\mu\hat{v}_\nu - (1+r)\hat{w}_\mu\hat{w}_\nu] \tag{6}$$

with fixed norm ($\frac{1}{3}\mathrm{Tr}AA^* = \Delta^2$) and $-1 \le r \le -\frac{1}{2}$, are nearly degenerate. This accidental degeneracy is exact in the uniform fourth-order Ginzburg-Landau (GL) theory.[7] Higher-order terms in the GL free energy stabilize the $r = -\frac{1}{2}$ state [Eq. (3)], but the difference in energy among the unitary states ("energy splitting") is never more than a few percent of the condensation energy, even at $T = 0$ K.[9] Consequently, there is a larger length, $\xi_r >> \xi_T$, at which the gradient energy is comparable to the energy splitting, but it is still small compared to the condensation energy. In the Ginzburg-Landau limit $\xi_r \gtrsim \xi_T/(1-T/T_c)^{1/2}$, but $\xi_r \gtrsim 10\xi_T$ even at low temperatures. At distances $\xi_T << \rho \lesssim \xi_r$ from the center of a vortex the order parameter may vary among the unitary states in order to minimize the sum of the gradient energy and energy splitting. The kinetic energy can be lowered by aligning $\hat{w}$ (the eigenvector with smallest eigenvalue) with $\vec{v}_s$ and allowing r to deviate from $-\frac{1}{2}$; however, for $r \neq -\frac{1}{2}$ the bending energy is nonzero since the vortex order parameter is no longer axial symmetric. In the intermediate region $\xi_T << \rho << \xi_r$, the competition between the kinetic and bending energies yields the minimum energy vortex

$$A_{\mu\nu}(\vec{R}) = N\Delta(T)e^{i\phi}[\hat{z}_\mu\hat{z}_\nu + r\hat{\rho}_\mu\hat{\rho}_\nu - (1+r)\hat{\phi}_\mu\hat{\phi}_\nu] , \qquad (7)$$

with $r = 6-\sqrt{43} \simeq -0.5574$ and $N = \{3/[1+r^2 + (1+r)^2]\}^{1/2} \simeq 1.41$; and corresponds to a 2π disgyration in addition to a 2π phase vortex.[6]

Near the core, $\rho \sim \xi_T$, the gradient energy is comparable to the condensation energy, so the order parameter will deviate from the unitary class if the total (condensation plus gradient) energy can be lowered. Richardson has studied the vortex solutions to the GL equations.[5] Although he does not include the higher-order terms in the GL free energy, his solutions should be valid for $\rho << \xi_r$ where the energy splitting is small compared with the gradient energy.

Richardson writes a general form of the vortex order parameter, valid for $0 \le \rho < \infty$, as

$$A_{\mu\nu} = \frac{\Delta(T)}{\sqrt{2}} e^{i\phi}\{[f_1\hat{\rho}_\mu\hat{\rho}_\nu + f_2\hat{\phi}_\mu\hat{\phi}_\nu - (f_1+f_2)\hat{z}_\mu\hat{z}_\nu] + ig(\hat{\rho}_\mu\hat{\phi}_\nu + \hat{\phi}_\mu\hat{\rho}_\nu)\} , \qquad (8)$$

where $f_1(\rho)$, $f_2(\rho)$, and $g(\rho)$ are solutions to the radial GL equations Eqs. (4.24) of Ref. 5. Since Richardson does not include the sixth-order free energy the asymptotic ($\rho \to \infty$) limit of his lowest-energy vortex

$$f_1 \to 1.112 , \quad f_2 \to 0.8832 , \quad g \to -0.9624(\xi_T/\rho)^2 \qquad (9)$$

is equivalent (up to a uniform phase factor) to Eq. (7). We conjecture the minimum-energy vortex solution, for all temperatures, interpolates smoothly between the local equilibrium state with $f_1 = f_2 = 1$ and $g = 0$ for $\rho >> \xi_r$ [equivalently Eq. (3)], and the state with $f_1 \neq f_2$ and $g \neq 0$ in Eqs. (8) and (9) for $\rho < \xi_r$.

Inside the core, $\rho << \xi_T$, the radial functions vanish linearly with ρ.[5] A smooth interpolation between the core ($\rho << \xi_T$), intermediate region ($\xi_T << \rho << \xi_r$), and asymptotic region ($\rho >> \xi_r$) is shown in Fig. 1.[10]

The interesting physical property of $^3P_2$ vortices follows from the deviations of the vortex order parameter [Eq. (8)] from the axial-symmetric unitary state in Eq. (3). In many respects the order parameter may be thought of as a local pair wave function. In particular we can expand this ampli-

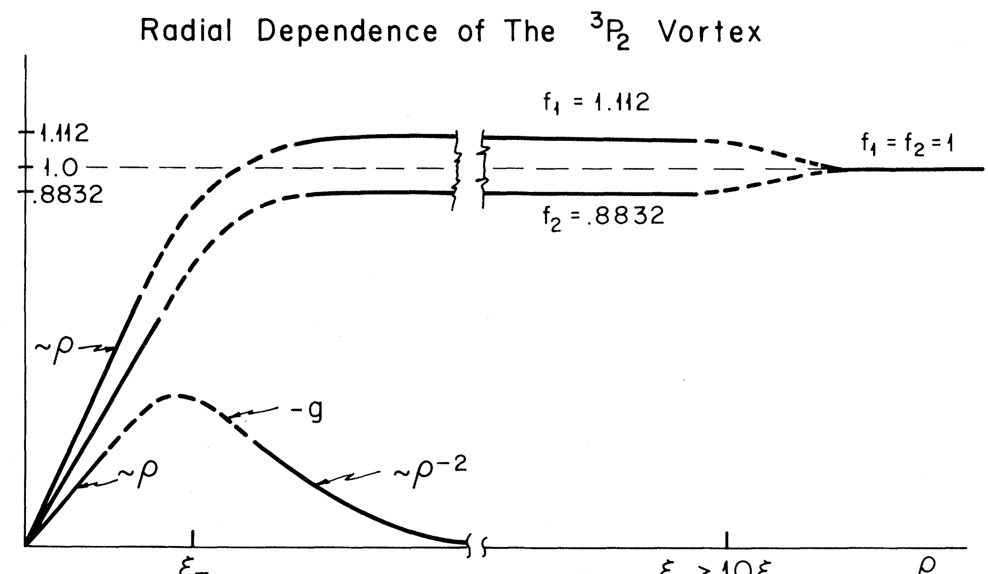

FIG. 1. Radial dependence of a $^3P_2$ vortex. The dashed lines interpolate between regions where the radial dependence of f_1, f_2, and g are known. The slopes for $\rho < \xi_T$ can only be determined by numerically integrating the Ginzburg-Landau equations.

tude $|\Psi\rangle$ as

$$|\Psi\rangle=\sum_{M_J=-2}^{+2}\psi_{M_J}\,|J=2,M_J\rangle\ . \tag{10}$$

The amplitudes ψ_{M_J} can be expressed in terms of $A_{\mu\nu}$ by inverting Eqs. (4) of Ref. 7. In the fixed coordinate system (x,y,z) these amplitudes are

$$\begin{aligned}&\psi_0\propto e^{i\phi}(f_1+f_2)\ ,\\&\psi_{\pm1}\equiv0\ ,\\&\psi_{\pm2}\propto e^{i\phi}e^{\pm2i\phi}[(f_1-f_2)\pm2g]\ .\end{aligned} \tag{11}$$

Thus, there is a spin polarization,

$$\begin{aligned}\langle\Psi|S_z|\Psi\rangle&\propto|\psi_{+2}|^2-|\psi_{-2}|^2\\&\propto g(f_1-f_2)\ ,\end{aligned} \tag{12}$$

that depends on both (f_1-f_2), the deviation of the unitary component from the axial-symmetric equilibrium state, and g, the deviation of the order parameter from the unitary class.

An estimate of the magnitude of the vortex spin polarization requires a microscopic calculation of the spin density

$$\begin{aligned}\vec{S}(\vec{R})&=\tfrac{1}{2}\hbar\vec{\sigma}(\vec{R})\\&=\tfrac{1}{2}\hbar\langle\psi^\dagger_\alpha(\vec{R})\vec{\sigma}_{\alpha\beta}\psi_\beta(\vec{R})\rangle\ ,\end{aligned} \tag{13}$$

which can be written in terms of the Fourier-transformed thermal Green's function

$$\vec{\sigma}(\vec{R})=T\sum_n\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}(\vec{\sigma}G(\vec{k},\epsilon_n;\vec{R}))\ . \tag{14}$$

To estimate $\vec{\sigma}(\vec{R})$ we use a local equilibrium approximation for G. When expanded to second order in $\hat{\Delta}(\hat{k};\vec{R})$, the superfluid correction to the normal-state Green's function becomes

$$\delta G_{\alpha\beta}(\vec{k},\epsilon_n;\vec{R})=-\frac{(\hat{\Delta}(\hat{k};\vec{R})\hat{\Delta}^\dagger(\hat{k};\vec{R}))_{\alpha\beta}}{(i\epsilon_n-\xi_{\vec{k}})(\epsilon_n{}^2+\xi_{\vec{k}}{}^2)}\ , \tag{15}$$

and Eq. (14) becomes

$$\begin{aligned}\vec{\sigma}(\vec{R})=&\int\frac{d\Omega_{\hat{k}}}{4\pi}\mathrm{tr}(\vec{\sigma}\hat{\Delta}(\hat{k};\vec{R})\hat{\Delta}^\dagger(\hat{k};\vec{R}))\\&\times T\sum_n\int_{-\epsilon_c}^{\epsilon_c}d\xi_{\vec{k}}N(\xi_{\vec{k}})\frac{i\epsilon_n+\xi_{\vec{k}}}{(\epsilon_n{}^2+\xi_{\vec{k}}{}^2)^2}\ ,\end{aligned} \tag{16}$$

where ϵ_c is the BCS cutoff. After doing the frequency sum and evaluating the $\xi_{\vec{k}}$ integral we obtain

$$\begin{aligned}\vec{\sigma}(\vec{R})=&\tfrac{1}{2}N'(0)\ln(1.13\epsilon_c/k_BT_c)\\&\times\int\frac{d\Omega_{\hat{k}}}{4\pi}\mathrm{tr}(\vec{\sigma}\hat{\Delta}(\hat{k};\vec{R})\hat{\Delta}^\dagger(\hat{k};\vec{R}))\ ,\end{aligned} \tag{17}$$

which for the vortex described by Eq. (8) becomes

$$\begin{aligned}\vec{\sigma}(\vec{R})=&\tfrac{1}{3}N'(0)\ln(1.13\epsilon_c/k_BT_c)\Delta(T)^2\\&\times g(\rho)[f_1(\rho)-f_2(\rho)]\hat{z}\ .\end{aligned} \tag{18}$$

The vortex magnetization $\vec{M}(\vec{R})=(\gamma_n\hbar/2)\ \vec{\sigma}(\vec{R})$ is nonvanishing in the core region, $0<\rho<\xi_r$, and has a magnitude determined by

$$\begin{aligned}M_0&=\frac{\gamma_n\hbar}{2}\frac{k_F{}^3}{3\pi^2}\left[\frac{\Delta(T)}{k_BT_F}\right]^2\\&\simeq-1.2\times10^{11}\ \mathrm{G}\end{aligned} \tag{19}$$

for $|\gamma_n\hbar/2|\simeq10^{16}\ \mathrm{G\,fm^3}$, $k_F=2.0\ \mathrm{fm}^{-1}$, $\Delta=0.8$ MeV.[11]

ELECTRON VELOCITY RELAXATION

Magnetic vortices in a rotating $^3P_2$ neutron superfluid provide a mechanism for electron velocity relaxation which is different than that proposed by Feibelman.[4] In addition to electrons scattering from neutron excitations in the vortex core, as Feibelman suggested for s-wave vortices, electrons also scatter from the vortex-core magnetization. These mechanisms are intrinsically different because the core magnetization is due to the condensate rather than the excitations.

To estimate the velocity relaxation time we treat the dilute array of magnetic vortices as fixed scattering centers which interact with the ultrarelativistic electrons through $H_{\mathrm{int}}=-\ e\vec{\gamma}\cdot\vec{A}(\vec{x})$, where $\vec{A}(\vec{x})=A(\rho)\hat{\phi}$, with

$$A(\rho)=\frac{1}{\rho}\int_0^\rho|\vec{M}(\rho')|\rho'd\rho'\ .$$

We solve for the electron distribution in the relaxation-time approximation assuming that the electron fluid and vortex array have an initial relative velocity; the distribution function is then used to calculate the relaxation time τ_g for the electron mass current $\vec{g}(t)$ from $\partial\vec{g}/\partial t\,|_{t=0}=-\vec{g}(0)/\tau_g$. The calculation is outlined in the Appendix from which we obtain

$$\frac{1}{\tau_g}=\frac{5}{27\pi^4}n_v\left[\frac{c\Delta}{k_F}\right]x^{-2/3}\left[\frac{(m_ec^2)(m_nc^2)(\gamma_e\hbar\gamma_n\hbar)^2}{(\hbar c)^6}\right], \tag{20}$$

where m_e (m_n) is the electron (neutron) mass, $\gamma_e\hbar=-1.2\times10^{-14}$ MeV/G is the electron gyromagnetic ratio, x is the ratio of number densities of electrons to neutrons, and n_v is the areal density of vortices determined by $n_v=2m_n\Omega/\pi\hbar$. The rotational period of the Crab (Vela) pulsar[12] is $P=2\pi/\Omega=0.033$ sec (0.089 sec) so that $n_v\sim10^5$ cm^{-2}. If we measure Δ in MeV, k_F in fm^{-1}, and P in sec, then

$$\tau_g\simeq1.26\times10^8\frac{k_Fx^{2/3}P}{\Delta}\text{ sec}. \tag{21}$$

Below we compare this result for τ_g with Feibelman's calculation of velocity relaxation due to electrons scattering from excitations in the $^3P_2$ vortex core. Although Feibelman assumed s-wave vortices, his result should be approximately correct for $^3P_2$ vortices. Presumably the only qualitative difference between the excitation spectrum for $^3P_2$ vortices and $^1S_0$ vortices is that the core magnetization of a $^3P_2$ vortex splits the vortex-core excited states by an energy $\gamma_n\hbar M\simeq10^{-6}$ MeV. But this energy is negligible compared with the core excitation energy $\Delta^2/\epsilon_F\sim10^{-2}$ MeV. Feibelman's Eq. (45) can be written in terms of the scaled variables in Eq. (21) as

$$\tau_F=2.94\times10^5\left[\frac{Px^{2/3}}{k_F}\right]\left[\frac{\Delta}{T_8}\right]\times\exp\left[4.41\frac{\Delta^2}{k_F{}^2T_8}\right]\text{ sec}, \tag{22}$$

where T_8 is the interior temperature in units of 10^8 K.

Table I shows τ_g,τ_F, and $\tau\equiv(\tau_g{}^{-1}+\tau_F{}^{-1})^{-1}$ for interior densities and a range of temperatures believed to span those for the Crab and Vela; the $^3P_2$ gap is taken from Ref. 2 with $\Delta=1.68k_BT_c$. The strong dependence of τ_F on both temperature and gap is due to the probability $e^{-\Delta^2/\epsilon_Fk_BT}$ that a neutron excitation is available for scattering. The relaxation time due to electrons scattering from the vortex magnetization is only weakly dependent on Δ and T: $\tau_g\propto\Delta(T)^{-1}$. For $T_8\simeq1$ scattering from core excitations dominates vortex magnetization scattering so that $\tau\simeq\tau_F\sim$ minutes, while for $T_8<<1$ the core excitation probability is vanish-

TABLE I. Relaxation times for electron-$^3P_2$ vortex scattering. The total relaxation time is calculated from $\tau^{-1}=\tau_g{}^{-1}+\tau_F{}^{-1}$. The electron concentrations are taken from unpublished calculations of R. Smith. These relaxation times were calculated using the Crab period $P=0.033$ sec. To obtain corresponding times for Vela, multiply by 2.7.

ρ (10^{14} g/cm^3)	k_F (fm^{-1})	Δ (MeV)	x	τ_g	τ_F	τ	T_8
					1.76 min	1.76 min	1.0
1.55	1.4	0.12	0.043	68.9 d	23.5 min	23.5 min	0.10
					3.02 d	2.89 d	0.01
					5.38 min	5.38 min	1.0
2.31	1.6	0.34	0.046	29.1 d	5.39 hr	5.35 hr	0.10
					3.7×10^5 yr	29.1 d	0.01
					10.1 min	10.1 min	1.0
3.29	1.8	0.55	0.050	21.4 d	2.86 d	2.53 d	0.1
					10^{15} yr	21.4 d	0.01
					18.8 min	18.8 min	1.0
4.52	2.0	0.79	0.057	18.0 d	6.39 d	4.72 d	0.1
					10^{27} yr	18.0 d	0.01
					24.4 min	24.4 min	1.0
6.01	2.2	0.92	0.068	19.2 d	1.75×10^2 d	17.3 d	0.1
					10^{31} yr	19.2 d	0.01
					30.8 min	30.8 min	1.0
7.81	2.4	1.04	0.084	21.3 d	3.68×10^2 d	20.1 d	0.1
					10^{33} yr	21.3 d	0.01

ingly small and $\tau_F \rightarrow \infty$ as $T_8 \rightarrow 0$. Since the core magnetization is associated with the condensate τ_g limits the relaxation time for $T_8 < 0.1$.

If we assume that the observed glitch relaxation times in the Crab pulsar, $\tau_d \simeq 4$ to 15 days,[12] are due to the velocity relaxation mechanisms discussed above, then these times suggest that the interior Crab temperature is $T \simeq 10^7$ K. A low Crab interior temperature requires an efficient cooling mechanism, presumably from β decay in the presence of pions or quarks.[13,14] A 2% fraction of pions will cool the interior to 10^7 K in the required 10^3 yr.[15] If a low interior temperature is ruled out and $T_8 \simeq 1$, then the $^3P_2$ gaps need to be 1.5 to 4 MeV. For low temperatures $T < 10^7$ K, the Feibelman time becomes unobservably long; however, τ_g limits the relaxation time at low temperature to 20 to 100 days (depending on the rotational period). Thus, glitches in older pulsars, if they occur, may have observable relaxation times.

The Vela temperature is expected to be lower than that of the Crab temperature, so that electron vortex scattering times are given by $\tau_g \simeq 50$ days. This relaxation time is consistent with the fast decay time scale between 30 and 150 days reported by Downs.[16] The nonexponential, long time ($\gtrsim 2$ yr) post-glitch behavior of Vela is probably due to a different mechanism.

COMMENTS

The relaxation times calculated above are rough estimates. Precise calculations require numerical solutions for the vortex structure and magnetization. The vortex structure discussed here was obtained from the Ginzburg-Landau theory, which is restricted to $T \lesssim T_c$. We expect the Ginzburg-Landau results to be qualitatively correct at low temperatures because $\xi_r >> \xi_T$ at all temperatures; numerical solutions to the Eilenberger equations for a $^3P_2$ vortex would test this assumption.

APPENDIX: ELECTRON SCATTERING FROM MAGNETIC VORTICES

To estimate the velocity relaxation time for electrons scattering from magnetic vortices we consider relativistic electrons scattering from a localized time-independent magnetic field. The transition probability per unit time is

$$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |M_{i \rightarrow f}|^2 \delta(\epsilon_f - \epsilon_i) . \quad \text{(A1)}$$

In the first Born approximation i and f are $\epsilon > 0$ solutions to the free-particle Dirac equation. These spinors are

$$\psi_{\vec{p}s}(\vec{x}) = \frac{1}{\sqrt{V}} e^{ip\cdot x/\hbar} \frac{1}{[2\epsilon(\epsilon+mc^2)]^{1/2}} \begin{pmatrix} (\epsilon+mc^2)\chi_s \\ c\vec{p}\cdot\vec{\sigma}\chi_s \end{pmatrix} , \quad \text{(A2)}$$

where $\chi_\pm$ are the two-component Pauli spinors $\binom{1}{0}$ and $\binom{0}{1}$. The matrix element is

$$M(\vec{p}s \rightarrow \vec{p}'s') = \int d^3x [\psi^\dagger_{\vec{p}'s'}(\vec{x}) H_{\text{int}} \psi_{\vec{p}s}(\vec{x})] , \quad \text{(A3)}$$

where

$$H_{\text{int}} = -e\vec{\gamma}\cdot\vec{A}(\vec{x}) = -e \begin{bmatrix} 0 & \vec{\sigma}\cdot\vec{A} \\ \vec{\sigma}\cdot\vec{A} & 0 \end{bmatrix} . \quad \text{(A4)}$$

After standard manipulations we write $M = M_c + M_\sigma$ with

$$M_c(\vec{p}s \rightarrow \vec{p}'s') = \frac{-ec}{2\epsilon V} \left[\int d^3x \, e^{i(\vec{p}-\vec{p}')\cdot\vec{x}/\hbar} (\vec{p}+\vec{p}')\cdot\vec{A}(\vec{x}) \right] \delta_{s,s'} , \quad \text{(A5)}$$

$$M_\sigma(\vec{p}s \rightarrow \vec{p}'s') = \left[\frac{-e\hbar}{2mc} \right] \left[\frac{mc^2}{\epsilon V} \right] \left[\int dx^3 e^{i(\vec{p}-\vec{p}')\cdot\vec{x}/\hbar} \vec{B}(\vec{x}) \right] \cdot \vec{\sigma}_{ss'} , \quad \text{(A6)}$$

and $\vec{B} = \vec{\nabla} \times \vec{A}$. The M_σ term represents the interaction of the electron spin magnetic moment with the field

$\vec{B}$. In the ultrarelativistic limit the current-field term M_c dominates the magnetic-moment interaction, which is significantly reduced (by mc^2/ϵ) compared with the nonrelativistic interaction.

For a cylindrically symmetric magnetic vortex with $\vec{B}=M(\rho)\hat{z}$ a convenient choice of gauge is $\vec{A}=A(\rho)\hat{\phi}$ with

$$A(\rho)=\frac{1}{\rho}\int_0^\rho \rho' M(\rho')d\rho' \; . \qquad \text{(A7)}$$

The calculation of M_c and M_σ can be reduced to one-dimensional integrals over the radial coordinate by introducing the projections of the momenta perpendicular to $\hat{z}$. In particular,

$$M_c=\left[\frac{-e\hbar}{2mc}\right]\left[\frac{mc^2}{\epsilon V}\right]2\pi\hbar\delta(p_z-p_z')\left[\frac{\text{sgn}(\cos\psi)\,|\vec{K}_\perp|}{i\hbar|\vec{q}_\perp|}\right]\pi(|\vec{q}_\perp|)\delta_{s,s'} \; , \qquad \text{(A8)}$$

$$M_\sigma=\left[\frac{-e\hbar}{2mc}\right]\left[\frac{mc^2}{\epsilon V}\right]2\pi\hbar\delta(p_z-p_z')\pi(|\vec{q}_\perp|)(\sigma_z)_{s,s'} \; , \qquad \text{(A9)}$$

where $\hbar\vec{q}_\perp=\vec{p}_\perp-\vec{p}_\perp'$, $\vec{K}_\perp=\vec{p}_\perp+\vec{p}_\perp'$, $p_z=\hat{z}\cdot\vec{p}$, $p_z'=\hat{z}\cdot\vec{p}'$, $\vec{p}_\perp=\vec{p}-p_z\hat{z}$, $\vec{p}_\perp'=\vec{p}'-p_z'\hat{z}$, $\cos\psi=\cos(\vec{p}_\perp,\vec{p}_\perp')$, and

$$\pi(q)=2\pi\int_0^\infty \rho M(\rho)J_0(q\rho)d\rho \; , \qquad \text{(A10)}$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind.

The electron velocity relaxation time is calculated from the electron distribution function which satisfies a kinetic equation[17]

$$\frac{\partial n_{\vec{p}s}}{\partial t}=N_v\sum_{\vec{p}',s'}\frac{2\pi}{\hbar}\delta(\epsilon-\epsilon')\,|M(\vec{p}s\to\vec{p}'s')|^2(n_{\vec{p}',s'}-n_{\vec{p},s}) \; , \qquad \text{(A11)}$$

where $N_v=n_vL^2$ is the number of vortex lines in the area L^2. The deviation of the distribution function from equilibrium

$$\delta n_{\vec{p}s}(t)=n_{\vec{p}s}(t)-n_0(\epsilon_{\vec{p}})$$
$$\simeq\delta(\epsilon_{\vec{p}}-\epsilon_F)\phi_{\hat{p}}(t)$$

is calculated in the relaxation time approximation. In particular, assume

$$\phi_{\hat{p}}(t)=\phi_{\hat{p}}(0)e^{-t/\tau} \; , \qquad \text{(A12)}$$

where the initial electron distribution $\phi_{\hat{p}}(0)$ is related to the initial relative velocity $\vec{u}$ between the electron fluid and vortex array by $\phi_{\hat{p}}(0)=p_F(\hat{p}\cdot\vec{u})$ with $\vec{u}\cdot\hat{z}=0$. The electron distribution relaxation time becomes $\tau^{-1}=\tau_c^{-1}+\tau_\sigma^{-1}$ with

$$\tau_c^{-1}=N_\tau\int_0^\pi d\psi\,\pi(|\vec{q}_\perp|)^2\cos^2(\psi/2) \; , \qquad \text{(A13)}$$

$$\tau_\sigma^{-1}=N_\tau\int_0^\pi d\psi\,\pi(|\vec{q}_\perp|)^2\sin^2(\psi/2) \; , \qquad \text{(A14)}$$

$$N_\tau=n_v\left[\frac{2\pi}{\hbar}\right]\left[\frac{mc^2}{\epsilon_F}\frac{\gamma_e\hbar}{2}\right]\frac{\epsilon_F}{(\pi\hbar c)^2} \; . \qquad \text{(A15)}$$

Further progress requires an analytic form for $\pi(q)$. We simplify the calculation by replacing $M(\rho)$ for a real $^3P_2$ vortex by the model function $\bar{M}\theta(\xi_T-\rho)$, where $\bar{M}$ is chosen so that

$$\bar{M}(\pi\xi_T^2)=2\pi\int_0^\infty \rho M(\rho)d\rho \; .$$

From Eqs. (18) and (19) $M(\rho)\propto M_0g\,(f_1-f_2)$. Thus, if we were to neglect the sixth-order free energy (equivalent to letting $\xi_r/\xi_T\to\infty$), then $M(\rho)\sim\rho^{-2}$ for $\rho\to\infty$. In this case the magnetic flux from a single vortex would diverge logarithmically with the size of the container. However, the sixth-order free energy introduces a natural cutoff; for $\rho>>\xi_r$: $M(\rho)=0$ because $f_1=f_2$. Although $M(\rho)$ is not known exactly, interpolation formulas give $\bar{M}\simeq M_0$. To be precise we use

$$M(\rho)=\tfrac{1}{5}M_0\theta(\xi_r-\rho)[\theta(\xi_T-\rho)(\rho/\xi_T)^2+\theta(\rho-\xi_T)(\xi_T/\rho)^2] \; ,$$

determined from Eqs. (9), (18), and (19), continuity at ξ_T, and $T_c/T_F\simeq10^{-3}$. An integration gives $\bar{M}\simeq M_0$ for $\xi_r/\xi_T\simeq10$.

With the model vortex Eq. (A10) becomes

$$\pi(q)=2\pi\bar{M}(\xi_T/q)J_1(q\xi_T) \; , \qquad \text{(A16)}$$

while Eqs. (A13) and (A14) give

$$\tau^{-1}=\tau_0^{-1}\left[\frac{k_{F,e}}{k_\perp}\right]^2 f(z)$$

with

$$\tau_0{}^{-1}=4\pi(n_v\xi_T{}^2)\left[\frac{m_ec^2(\frac{1}{2}\gamma_e\hbar)M_0}{\epsilon_{F,e}}\right]^2(ck_{F,e}) , \tag{A17}$$

$$f(z)=\int_0^1 dx\frac{[J_1(zx)]^2}{x^2(1-x^2)^{1/2}} , \tag{A18}$$

$$z=2k_\perp\xi_T . \tag{A19}$$

The electron Fermi wave vector and energy are related by $\epsilon_{F,e}\simeq\hbar ck_{F,e}$ in the ultrarelativistic limit. The perpendicular component of the electron wave vector is $k_\perp=k_{F,e}\sin\theta$ where θ is measured from the $\hat{z}$ direction. Consequently $z>>1$ except for electrons which are at grazing incidence relative to the vortex line. Approximate formulas for $f(z)$ can be obtained,[18]

$$f(z)\simeq\begin{cases}\frac{\pi}{8}z^2, & z<<1\\ 0.44z+O(1), & z>>1 ,\end{cases} \tag{A20}$$

and a useful interpolation formula is

$$f(z)\simeq\frac{\pi}{8}z^2\left[\frac{1}{1+0.9z}\right] . \tag{A21}$$

To complete the calculation, we define the electron velocity relaxation time by the initial rate of decay of the electron current

$$\vec{g}(t)=2\int\frac{d^3p}{(2\pi\hbar)^3}\vec{p}\left[\frac{-\partial n_0}{\partial\epsilon_{\vec{p}}}\right]\phi_{\hat{p}}(t) . \tag{A22}$$

Then $\partial\vec{g}/\partial t\,|_{t=0}\equiv\vec{g}(t=0)/\tau_g$ gives

$$\tau_g{}^{-1}=\tfrac{3}{2}\tau_0{}^{-1}\int_{-1}^{1}\frac{d(\cos\theta)}{2}f(2k_{F,e}\xi_T\sin\theta) . \tag{A23}$$

To leading order in $(2k_{F,e}\xi_T)^{-1}<<1$, the θ integral gives

$$\tau_g{}^{-1}=\frac{5\pi^3}{12}(k_{F,e}\xi_T)(n_v\xi_T{}^2)\left[\frac{m_ec^2\gamma_e\hbar M_0}{2\epsilon_{F,e}}\right](ck_{F,e}) . \tag{A24}$$

Equation (20) is obtained from Eqs. (A24), (19), $k_{F,e}\simeq x^{1/3}k_F$, $\xi_T=(2/\pi)(\epsilon_F/\Delta)k_F{}^{-1}$, and $\epsilon_F=k_BT_F=(\hbar c)^2k_F{}^2/2m_n$.

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[10]Although we do not have exact solutions for the radial functions, we remark that a model calculation gives a lower energy for the magnetic vortex than the nonmagnetic vortex with $f_1(\rho)=f_2(\rho)$, $g(\rho)\equiv0$, and $\lim_{\rho\to\infty}f_1,f_2\to1$. In particular, by assuming the radial functions in the core are (1) linear in ρ, and (2) match the known Richardson solution at the core radius, which is used as a variational parameter, we find that the magnetic vortex has a lower energy provided the degeneracy length satisfies $\xi_r/\xi_T>24$. Our best estimate of this ratio, obtained by equating the energy splitting and the gradient energy, is $\xi_r/\xi_T\simeq60$.

[11]Our result for the vortex magnetization [Eq. (17)] depends on the derivative of the density of states at the Fermi energy, $N'(0)$, and the BCS cutoff ϵ_c. In arriving at our estimate of the scale of the vortex magnetization [Eq. (19)] we assume (i) a parabolic density of states, so that $N'(0)=\frac{1}{2}N(0)/k_BT_F$, and (ii) $\epsilon_c\propto k_BT_F$ because ϵ_c is roughly the scale for energy dependence of the normal-state pairing interaction. We do not believe these assumptions cause serious errors in our calculations. In fact, we expect the largest uncertainty in our estimates of the magnetic flux (see the Appendix) to be the difference between the exact radial dependence of f_1, f_2, and g and our interpolated functions shown in Fig. 1.

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