Topological Hall Effects in Magnetic Metals with Chiral Spin Textures

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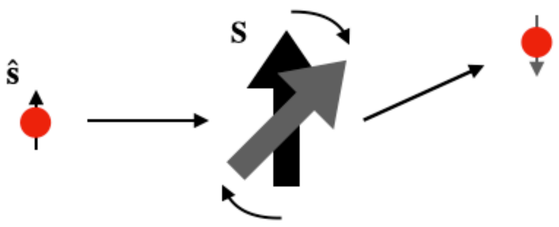
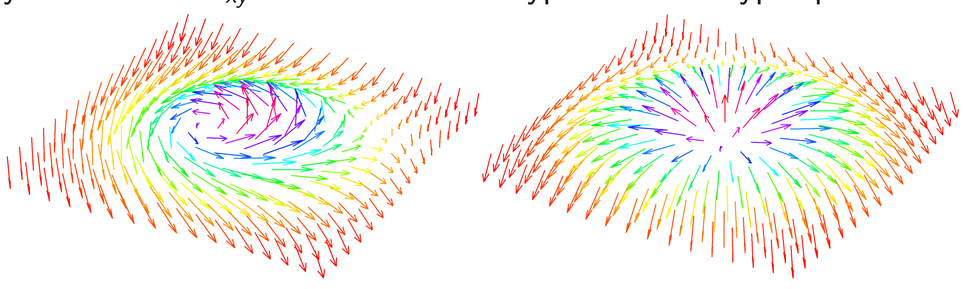
Introduction

- Magnetic metals have an exchange interaction between the conduction electron spin and localized one constructing the magnetization [1].
- *s-d* exchange interaction:



$$\mathcal{H}_{sd}(t) = J\sum_{\alpha\beta}\int d^3r\psi^\dagger_\alpha(\mathbf{r})(\hat{\sigma})_{\alpha\beta}\cdot\mathbf{S}(\mathbf{r},t)\psi_\beta(\mathbf{r}).$$

- The weak *s-d* exchange interaction causes the topological Hall conductivity due to the spin chirality [2,3], whose spatial average is proportional to the topological charge, $N_{ij} = (4\pi S^3)^{-1}\int dr_i dr_j \mathbf{S}\cdot(\partial_{r_i}\mathbf{S}\times\partial_{r_j}\mathbf{S})$ [4].
- 2D skyrmions with $N_{xy} = 1$ called Bloch-type and Néel-type spin texture:



- We derive a Boltzmann transport equation with the weak *s-d* exchange interaction from the Kondo Hamiltonian and Dyson equation.
- We calculate charge, spin, and heat transport in magnetic metals including both the static chiral magnets and steady-state motion of 2D skyrmions.

Boltzmann transport equation

- Right and left Dyson equations in the gauge-covariant Wigner representation:

$$\left[(\varepsilon-\xi_p)\check{1} - J\check{\boldsymbol{\sigma}}\cdot\mathbf{S}(\mathbf{r},t) - \check{\Sigma}(\mathbf{r};\varepsilon,t)\right] * \check{G}(\mathbf{p},\mathbf{r};\varepsilon,t) = \check{1},$$
$$\check{G}(\mathbf{p},\mathbf{r};\varepsilon,t) * \left[(\varepsilon-\xi_p)\check{1} - J\check{\boldsymbol{\sigma}}\cdot\mathbf{S}(\mathbf{r},t) - \check{\Sigma}(\mathbf{r};\varepsilon,t)\right] = \check{1},$$

  $\xi_p = p^2/2m - \mu$: single particle energy for an ideal gas.
- Moyal product $*$ [5] & Keldysh matrices:

$$\check{a}(\mathbf{p},\mathbf{r};\varepsilon,t) * \check{b}(\mathbf{p},\mathbf{r};\varepsilon,t) \equiv \check{a}(\mathbf{p},\mathbf{r};\varepsilon,t)\mathrm{e}^{\frac{i\hbar}{2}(\overleftarrow{\partial}_\mathbf{r}\cdot\overrightarrow{\partial}_\mathbf{p} - \overleftarrow{\partial}_t\overrightarrow{\partial}_\varepsilon - \overleftarrow{\partial}_\mathbf{p}\cdot\overrightarrow{\partial}_\mathbf{r} + \overleftarrow{\partial}_\varepsilon\overrightarrow{\partial}_t)}$$
$$\times\, \mathrm{e}^{\frac{i\hbar}{2}\left[-e\mathbf{E}\cdot(\overleftarrow{\partial}_\mathbf{p}\overrightarrow{\partial}_\varepsilon - \overleftarrow{\partial}_\varepsilon\overrightarrow{\partial}_\mathbf{p}) + \frac{e}{c}\mathbf{B}\cdot(\overleftarrow{\partial}_\mathbf{p}\times\overrightarrow{\partial}_\mathbf{p})\right]}\check{b}(\mathbf{p},\mathbf{r};\varepsilon,t).$$

$$\check{\boldsymbol{\sigma}} \equiv \begin{bmatrix}\hat{\boldsymbol{\sigma}} & \hat{0}\\ \hat{0} & \hat{\boldsymbol{\sigma}}\end{bmatrix}, \quad (\hat{\boldsymbol{\sigma}})_{\alpha\beta} \equiv (\hat{\sigma}_x)_{\alpha\beta}\hat{\mathbf{x}} + (\hat{\sigma}_y)_{\alpha\beta}\hat{\mathbf{y}} + (\hat{\sigma}_z)_{\alpha\beta}\hat{\mathbf{z}}, \quad \hat{\sigma}_i : \text{Pauli matrices.}$$

$$\check{\Sigma} = n_a|U|^2\int\frac{d^3p}{(2\pi\hbar)^3}\check{G},\ n_a, U : \text{Nonmagnetic impurity density and potential.}$$

- The right Dyson equation is subtracted from the left one or added to the left one, and retained up to the 1st-order in the gradient expansion for the space and momentum.

$$\left[\varepsilon\check{1} - J\check{\boldsymbol{\sigma}}\cdot\mathbf{S} - \check{\Sigma}, \check{G}\right]_\circ + i\hbar\mathbf{v}_\mathbf{p}\cdot\frac{\partial\check{G}}{\partial\mathbf{r}} - \frac{i\hbar}{2}\left\{\frac{\partial J\check{\boldsymbol{\sigma}}\cdot\mathbf{S}}{\partial\mathbf{r}} + \frac{\partial\check{\Sigma}}{\partial\mathbf{r}}, \frac{\partial\check{G}}{\partial\mathbf{p}}\right\}$$
$$+ i\hbar e\mathbf{v}_\mathbf{p}\cdot\mathbf{E}\frac{\partial\check{G}}{\partial\varepsilon} + i\hbar e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\mathbf{p}\times\mathbf{B}\right)\cdot\frac{\partial\check{G}}{\partial\mathbf{p}} = \check{0},$$

$$\frac{1}{2}\left\{(\varepsilon-\xi_p)\check{1} - J\check{\boldsymbol{\sigma}}\cdot\mathbf{S} - \check{\Sigma}, \check{G}\right\}_\circ - \frac{i\hbar}{4}\left[\frac{\partial J\check{\boldsymbol{\sigma}}\cdot\mathbf{S}}{\partial\mathbf{r}} + \frac{\partial\check{\Sigma}}{\partial\mathbf{r}}, \frac{\partial\check{G}}{\partial\mathbf{p}}\right] = \check{1},$$

$$\check{a}(\varepsilon,t)\circ\check{b}(\varepsilon,t) \equiv \check{a}(\varepsilon,t)\mathrm{e}^{\frac{i\hbar}{2}(\overleftarrow{\partial}_\varepsilon\overrightarrow{\partial}_t - \overleftarrow{\partial}_t\overrightarrow{\partial}_\varepsilon)}\check{b}(\varepsilon,t).$$

- Quasiclassical Green's functions: [6]

$$\check{\mathfrak{G}}(\hat{\mathbf{p}},\mathbf{r};\varepsilon,t) \equiv \mathrm{P}\int_{-\infty}^{\infty}d\xi_p\check{G}(\mathbf{p},\mathbf{r};\varepsilon,t) \equiv \begin{bmatrix}\hat{\mathfrak{g}}^\mathrm{R} & \hat{\mathfrak{g}}^\mathrm{K}\\ \hat{0} & \hat{\mathfrak{g}}^\mathrm{A}\end{bmatrix},$$
$$\check{\mathfrak{G}}^{(1)}(\hat{\mathbf{p}},\mathbf{r};\varepsilon,t) \equiv \mathrm{P}\int_{-\infty}^{\infty}d\xi_p\left[\xi_p\check{G}(\mathbf{p},\mathbf{r};\varepsilon,t) + \check{1}\right] \equiv \begin{bmatrix}\hat{\mathfrak{g}}^{\mathrm{R}(1)} & \hat{\mathfrak{g}}^{\mathrm{K}(1)}\\ \hat{0} & \hat{\mathfrak{g}}^{\mathrm{A}(1)}\end{bmatrix}.$$

- We consider the weak nonmagnetic impurity scattering, $\hbar/\tau \ll |J|S$, and assume the spectral function is the same form as Fermi liquid, i.e. $\hat{\mathfrak{g}}^\mathrm{R} \approx -i\pi\hat{1}$ and $\hat{\mathfrak{g}}^\mathrm{A} \approx i\pi\hat{1}$.

$$\frac{\partial\hat{\mathfrak{g}}^\mathrm{K}}{\partial t} + \mathbf{v}_\mathbf{p}\cdot\frac{\partial\hat{\mathfrak{g}}^\mathrm{K}}{\partial\mathbf{r}} + e\mathbf{v}_\mathbf{p}\cdot\mathbf{E}\frac{\partial\hat{\mathfrak{g}}^\mathrm{K}}{\partial\varepsilon} + e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\mathbf{p}\times\mathbf{B}\right)\cdot\frac{\partial\hat{\mathfrak{g}}^\mathrm{K}}{\partial\mathbf{p}}$$
$$+ \frac{i}{\hbar}\left[J\hat{\boldsymbol{\sigma}}\cdot\mathbf{S}, \hat{\mathfrak{g}}^\mathrm{K}\right]_\circ - \frac{1}{2}\left\{\frac{\partial J\hat{\boldsymbol{\sigma}}\cdot\mathbf{S}}{\partial\mathbf{r}}, \frac{\partial\hat{\mathfrak{g}}^\mathrm{K}}{\partial\mathbf{p}}\right\} = -\frac{1}{\tau}\left(\hat{\mathfrak{g}}^\mathrm{K} - \int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\hat{\mathfrak{g}}^\mathrm{K}\right),$$
$$\hat{\mathfrak{g}}^{\mathrm{K}(1)} = \varepsilon\hat{\mathfrak{g}}^\mathrm{K} - \frac{1}{2}\left\{J\hat{\boldsymbol{\sigma}}\cdot\mathbf{S}, \hat{\mathfrak{g}}^\mathrm{K}\right\}_\circ, \qquad \frac{\hbar}{\tau} = 2\pi n_a N_f|U|^2.$$

Linear response of static magnetic metals

- Charge, spin, and thermal current density and magnetization:

$$\mathbf{j} = eN_f\mathrm{Tr}\int_{-\varepsilon_\mathrm{c}}^{\varepsilon_\mathrm{c}}\frac{d\varepsilon}{4\pi i}\int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\mathbf{v}_\mathbf{p}\left[\hat{\mathfrak{g}}^\mathrm{K} + \frac{N_f'}{N_f}\hat{\mathfrak{g}}^{\mathrm{K}(1)}\right],$$
$$\mathbf{j}_\mathrm{s}^{(i)} = eN_f\mathrm{Tr}\int_{-\varepsilon_\mathrm{c}}^{\varepsilon_\mathrm{c}}\frac{d\varepsilon}{4\pi i}\int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\hat{\sigma}_i\mathbf{v}_\mathbf{p}\left[\hat{\mathfrak{g}}^\mathrm{K} + \frac{N_f'}{N_f}\hat{\mathfrak{g}}^{\mathrm{K}(1)}\right],$$
$$\mathbf{j}_Q = N_f\mathrm{Tr}\int_{-\varepsilon_\mathrm{c}}^{\varepsilon_\mathrm{c}}\frac{d\varepsilon}{4\pi i}\int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\varepsilon\mathbf{v}_\mathbf{p}\left[\hat{\mathfrak{g}}^\mathrm{K} + \frac{N_f'}{N_f}\hat{\mathfrak{g}}^{\mathrm{K}(1)}\right],$$
$$\mathbf{M} = -\mu_\mathrm{B}N_f\mathrm{Tr}\int_{-\varepsilon_\mathrm{c}}^{\varepsilon_\mathrm{c}}\frac{d\varepsilon}{4\pi i}\int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\hat{\boldsymbol{\sigma}}\left[\hat{\mathfrak{g}}^\mathrm{K} + \frac{N_f'}{N_f}\hat{\mathfrak{g}}^{\mathrm{K}(1)}\right] - \frac{2\mu_\mathrm{B}\mathbf{S}}{\hbar}.$$

  N_f': Derivative of the normal density of states at the Fermi energy.
- Topological electrical and thermal Hall effects:

$$\mathbf{j} = (\overleftrightarrow{\sigma}^{(\mathrm{n})} + \overleftrightarrow{\sigma}^{(\mathrm{a})})\mathbf{E},\ \overleftrightarrow{\sigma}^{(\mathrm{n})} : \text{Drude conductivity},\ \overleftrightarrow{\sigma}^{(\mathrm{a})} = \frac{4\tau^4e^2N_f'v_f^2J^3S^3}{3\hbar m}\overleftrightarrow{\chi}^{(\mathrm{s})},$$
$$\mathbf{j}_Q = -\frac{\pi^2}{3}\left(\frac{k_\mathrm{B}}{e}\right)^2(\overleftrightarrow{\sigma}^{(\mathrm{n})} + \overleftrightarrow{\sigma}^{(\mathrm{a})})T\frac{\partial T}{\partial\mathbf{r}},\ \chi_{ij}^{(\mathrm{s})} \equiv \mathbf{n}\cdot\left(\frac{\partial\mathbf{n}}{\partial r_i}\times\frac{\partial\mathbf{n}}{\partial r_j}\right),\ \mathbf{S} = \mathbf{n}S.$$

- Spin Hall and Nernst effects:

$$\mathbf{j}_\mathrm{s}^{(i)} = \overleftrightarrow{\sigma}_\mathrm{s}^{(i)}\mathbf{E},\ \overleftrightarrow{\sigma}_\mathrm{s}^{(i)} = -\frac{4\tau^4e^2N_fv_f^2J^2S^2}{3\hbar m}\overleftrightarrow{\chi}^{(i)},\ \chi_{jk}^{(i)} \equiv \left(\frac{\partial\mathbf{n}}{\partial r_j}\times\frac{\partial\mathbf{n}}{\partial r_k}\right)_i,$$
$$\mathbf{j}_\mathrm{s}^{(i)} = -S^{(\mathrm{n})}\overleftrightarrow{\sigma}_\mathrm{s}^{(i)}\frac{\partial T}{\partial\mathbf{r}},\ S^{(\mathrm{n})} = \frac{\pi^2}{3}\frac{k_\mathrm{B}^2T}{e}\frac{N_f'}{N_f} : \text{Normal Seebeck coefficient.}$$

- Edelstein effect and temperature-gradient-induced magnetization:

$$\mathbf{M} = \overleftrightarrow{\Upsilon}\mathbf{E},\ \overleftrightarrow{\Upsilon} = -\frac{4\tau^3e\mu_\mathrm{B}N_fJ^2S^2}{\hbar m}\overleftrightarrow{\chi}^{(\mathrm{v})},\ \chi_{ij}^{(\mathrm{v})} \equiv \left(\mathbf{n}\times\frac{\partial\mathbf{n}}{\partial r_j}\right)_i,\ \mathbf{M} = -S^{(\mathrm{n})}\overleftrightarrow{\Upsilon}\frac{\partial T}{\partial\mathbf{r}}.$$

Steady-state motion of 2D skyrmions

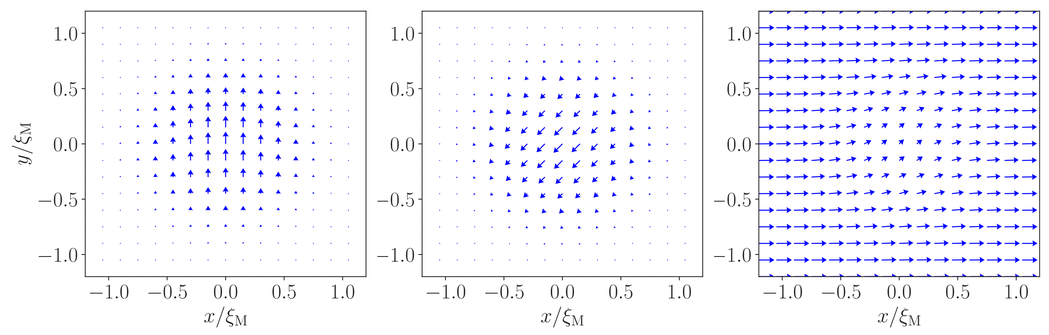
- We assume the form of localized spin with small velocity $\mathbf{v}_\mathrm{S}$ as $\mathbf{S}(\mathbf{r},t) = \mathbf{S}(\mathbf{r}-\mathbf{v}_\mathrm{S}t)$, and the form of $\mathbf{v}_\mathrm{S}$ as

$$\mathbf{v}_\mathrm{S} = -\frac{\tau e\tilde{\rho}}{m}\mathfrak{R}(\tilde{\varphi})\mathbf{j},\quad \mathfrak{R}(\tilde{\varphi}) \equiv \begin{bmatrix}\cos\tilde{\varphi} & -\sin\tilde{\varphi}\\ \sin\tilde{\varphi} & \cos\tilde{\varphi}\end{bmatrix} : \text{2D rotation matrix.}$$

- Conductivity due to the skyrmion motion:

$$\mathbf{j} = (\overleftrightarrow{\sigma}^{(\mathrm{n})} + \overleftrightarrow{\sigma}^{(\mathrm{a})} + \overleftrightarrow{\sigma}^{(\mathrm{s})})\mathbf{E},\quad \overleftrightarrow{\sigma}^{(\mathrm{s})} = \tilde{\rho}\sin\tilde{\varphi}\sigma_{xy}^{(\mathrm{a})}\overleftrightarrow{\sigma}^{(\mathrm{n})} - \tilde{\rho}\cos\tilde{\varphi}\sigma_{xx}^{(\mathrm{n})}\overleftrightarrow{\sigma}^{(\mathrm{a})}.$$

- The left and center figures are the emergent field without the skyrmion motion and induced by the skyrmion motion ($\tilde{\varphi} = -\pi/4$), respectively, and right figure is the electric field with all contributions including the Drude term.



- The same term does not contribute to $\mathbf{j}_Q$. The contribution to $\mathbf{j}_Q$ from the skyrmion motion comes from higher-order correction terms.
- Lorenz ratio tensor:

$$L_{ij} \equiv \frac{\kappa_{ij}}{\sigma_{ij}T} = \frac{\pi^2}{3}\left(\frac{k_\mathrm{B}}{e}\right)^2\frac{\sigma_{ij}^{(\mathrm{n})} + \sigma_{ij}^{(\mathrm{a})}}{\sigma_{ij}^{(\mathrm{n})} + \sigma_{ij}^{(\mathrm{a})} + \sigma_{ij}^{(\mathrm{s})}}.$$ Violaiton of Wiedemann-Franz law

Conclusion

- The Wiedemann-Franz law and Mott's relation hold even for the anomalous terms for scattering off static skyrmions.
- Conduction electron scattering off moving skyrmions leads to violaiton of the Wiedemann-Franz law.

Acknowledgements & References

This research was supported by the Center for Computation and Technology at Louisiana State University, and the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Superconducting Quantum Materials and Systems Center (SQMS) under contract number DE-AC02-07CH11359.

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