Electromagnetic Response of Disordered Superconducting Cavities [1]

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Introduction

- We present results for the resonant frequency shift and quality factor of disordered Nb SRF cavities driven out of equilibrium by the resonant microwave field.
- The theory is based on the nonequilibrium theory of superconductivity for the current response to the electromagnetic (EM) field at the vacuum-metal interface.
- We accurately predict the observed frequency shifts with a precision of order several Hz over the full temperature range $0 \lesssim T \leq T_c$, including the negative frequency shift anomalies that are observed very near T_c.
- The origin of these anomalies is shown to be the competition between the normal metal skin depth and the London penetration depth which diverges as $T \to T_c^-$. An analytic approximation to the full current response, valid for $|T - T_c| \ll T_c$, accounts for the negative frequency shift near T_c.
- The non-monotonic dependence of the quality factor on the quasiparticle scattering rate is related to the pair-breaking effect of disorder on the superfluid fraction, and thus the London penetration depth.

Current Response of an SRF Cavity

Solve Maxwell's equations for the EM field, including the screening current response to a transverse EM field within the superconductor,

$$\vec{J}(\vec{r},t) = -\frac{c}{4\pi}\int_{-\infty}^{t} K(t-t')\,\vec{A}(\vec{r},t')\,, \tag{1}$$

$K^R(t-t') \equiv \Theta(t-t')\,K(t-t')$ is the retarded response function.
We assume there is sufficient disorder that the current response to $\vec{A}(\vec{r},t)$ can be evaluated in *the local limit* $K(\omega,q) \approx K(\omega)$ and Ampère-Maxwell equation for the EM field in the superconductor:

$$\left(\nabla^2 + \frac{\omega^2}{c^2} - K(\omega)\right)\vec{A}(\vec{r},\omega) = 0\,, \tag{2}$$

where $K(\omega)$ is the Fourier transform of the retarded current response function appearing in Eq. (1).
Inside the cavity : the wave equation in vacuum $K(\omega) = 0$.
Solve the field wave equation with the *boundary condition for the continuity of* $\vec{A}$ *and the normal derivative at the vacuum-superconductor interface.*

Keldysh Response Function

The EM linear response function $K(\vec{q},\omega)$ for superconductors is calculated using Keldysh's formulation of nonequilibrium response in the quasiclassical approximation. In local limit, we can set $q = 0$ and the current response function is given by [2]

$$\begin{aligned}
&K(\omega;\tau,T) = \frac{4\pi\omega}{ic^2}(\sigma_1 + i\sigma_2) = \frac{\pi\sigma_{\rm D}}{ic^2\tau}\int_{-\infty}^{+\infty} d\epsilon I(\epsilon,\omega;\tau,T)\\
&I(\epsilon,\omega;\tau,T) \equiv \Big\{ \tanh\left(\frac{\epsilon-\omega/2}{2T}\right)\frac{1}{D_+^R + D_-^R + 1/\tau}\left(\frac{\epsilon^2+\Delta^2-\omega^2/4}{D_+^R D_-^R}+1\right)\\
&- \tanh\left(\frac{\epsilon+\omega/2}{2T}\right)\frac{1}{D_+^A + D_-^A + 1/\tau}\left(\frac{\epsilon^2+\Delta^2-\omega^2/4}{D_+^A D_-^A}+1\right)\\
&+ \left[\tanh\left(\frac{\epsilon+\omega/2}{2T}\right) - \tanh\left(\frac{\epsilon-\omega/2}{2T}\right)\right]\frac{1}{D_+^R + D_-^A + 1/\tau}\left(\frac{\epsilon^2+\Delta^2-\omega^2/4}{D_+^R D_-^A}+1\right)\Big\}\\
&D_\pm^{R/A} \equiv \sqrt{\Delta^2 - (\epsilon \pm \omega/2 \pm i\delta)^2}
\end{aligned} \tag{3}$$

where $\sigma_{\rm D} = \frac{2}{3}N_f e^2 v_f^2 \tau = ne^2\tau/m^*$ is the Drude result for the d.c. conductivity for charge e quasiparticles, with Fermi velocity v_f, Fermi momentum p_f, density of states, $N_f = \frac{3}{2}n/p_f v_f$, where n is the electron density and $m^* = p_f/v_f$ is the quasiparticle effective mass.
At $\Delta \to 0$ Eq. (3) results in the normal state conductivity

$$\sigma_n(\omega) = \frac{\sigma_{\rm D}}{1 - i\omega\tau}\,. \tag{4}$$

Superconductivity leads to significant changes in conductivity and consequently in the cavity resonance frequency and the quality factor, that are sensitive to disorder and temperature. We analyze and predict experimental data using Eqs. (2), (3) and boundary conditions for the EM field at the vacuum-metal interface (see Eq. (6)).

Material parameters for pure Niobium

The current response depends on the *frequency and wavevector*, ω and $\vec{q}$, of the electromagnetic field within the metal, includes the condensate response (supercurrent) and the dissipative response of unbound quasiparticles scattered by the random potential with *scattering rate*, $1/\tau$.
The other important internal timescale is *the pair formation time*, $\tau_0 \equiv \hbar/2\pi k_B T_c$, and the corresponding ballistic *pair correlation length* is $\xi_0 = v_f\tau_0 = \hbar v_f/2\pi k_B T_c$. For pure Nb the pair correlation length and *London penetration depth* are comparable, $\xi_0 \approx \lambda_{\rm L_0} \simeq 33\,$nm.
We consider the long-wavelength ("local") limit, $q \leq 1/\lambda_{\rm L} \ll 1/\xi_0$, for the current response, which is achieved in Nb when the mean scattering time is comparable to the pair formation time, $\tau \sim \tau_0$.
This level of disorder leads to weak suppression of T_c, a reduction in the superconducting coherence length, ξ, and an increase in the London penetration depth such that $\lambda \gg \xi$.
However, disorder leads to a finite mean free path for quasiparticles, $\ell = v_f\tau$, converting ballistic motion to diffusive motion, and thus a reduction in the pair correlation length. Disorder is also pair breaking, particularly in the presence of screening currents, resulting in a reduction of the superfluid density, n_s. Thus, the London penetration depth, $\lambda_{\rm L} \propto 1/\sqrt{n_s}$, increases due to scattering by the random potential.

T_c [K]	v_f [10^8 cm/s]	τ_0 [ps]	ξ_0 [nm]	$\lambda_{\rm L_0}$ [nm]	Δ_0 [meV]
9.33	0.257	0.131	33.0	33.0	1.55

Table 1: Material parameters for pure Niobium.

$f[GHz]$	$\tau[ps]$	$\ell[nm]$	τ/τ_0	$R_n[m\Omega]$	$\tau_{\rm R_n}[ps]$
0.65	0.173	45.092	1.327	4.37	0.184
1.30	0.224	58.235	1.713	5.45	0.236
2.60	0.257	66.700	1.963	7.10	0.279
3.90	0.249	64.730	1.905	8.96	0.262

Table 2: Material parameters for different cavities. The second column is the scattering time that gives the best fit of theory to experiment. The values of R_n were provided by D. Bafia [private communication].

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Cylindrical cavities

SRF cavities for accelerator applications adopt the *Tesla geometry* with axially symmetry. For simplicity of applying boundary conditions we consider *cylindrical cavities* of radius R and length L with geometric factor G

$$G = Z_0\,(\omega/c)\,\frac{\int_V d^3r\,|\vec{B}(\vec{r},\omega)|^2}{\int_S dS\,|\vec{B}(\vec{r}_S,\omega)|^2} = Z_0\,\frac{x_{01}}{2}\,\frac{L}{L+R}\,, \tag{5}$$

where $Z_0 = 4\pi/c \approx 377\,\Omega$ is the vacuum impedance. We correct our result obtained using the $\rm TM_{010}$ mode of the cylindrical cavity by replacing the value of G for the cylindrical cavity with the geometric factor computed for the lowest TM mode of the Tesla cavity.
$\rm TM_{010}$ is the lowest frequency of the cylindrical cavity with $\vec{A}(\vec{r},\omega) = A_0\,J_0(\omega\rho/c)\,\hat{z}$ in the gauge $\nabla\cdot\vec{A} = 0$. The Fourier component of the electric field is axial, $\vec{E}(\vec{r},\omega) = (i\omega/c)\,A_0\,J_0(\omega\rho/c)\,\hat{z}$, and the magnetic field circulates azimuthally, $\vec{B}(\vec{r},\omega) = A_0\,(\omega/c)\,J_1(\omega\rho/c)\,\hat{\varphi}$.
The penetration of the field over distance scales of order $\lambda_{\rm L} \ll R$ leads to a small shift in the resonance frequency, $\delta\omega$, which depends on the material properties of the superconductor via the current response function $K(\omega)$. Penetration of the electric field into the superconductor also leads to dissipation of the microwave field by quasiparticles scattering from the interface and from the random distribution of impurities, thus limiting the quality factor, Q.
Both $\delta\omega$ and Q are determined by an eigenvalue equation obtained from the boundary conditions on the vector potential and its derivative at the vacuum-superconductor interface. For the $\rm TM_{010}$ mode on the vacuum side, $A(\rho) = A_0\,J_0(\frac{\omega}{c}\rho)$. Thus, for $\rho \geq R$ we obtain $A(\rho) = A_0'\,\exp\{-\Lambda(\omega)(\rho - R)\}$, where $\Lambda(\omega) \equiv \sqrt{K(\omega) - (\omega/c)^2}$.
The ratio of the two boundary conditions reduces to

$$\Lambda(\varpi) = \sqrt{K(\varpi) - (\varpi/c)^2} = \left(\frac{\varpi}{c}\right)\frac{J_1(\varpi R/c)}{J_0(\varpi R/c)}\,, \tag{6}$$

where $J_1(x) = -dJ_0(x)/dx$. In this equation the complex eigenvalue defined by the solution of Eq. (6) is written as

$$\varpi \equiv \omega + \delta\omega - i\omega/2Q \tag{7}$$

where $\delta\omega$ is the frequency shift and Q is the quality factor.

Impedance, Resistance and Reactance

For SRF cavities the surface impedance $Z(\omega)$ resulting from the response of screening currents in the superconductor is defined by the ratio of the tangential fields at the interface, Thus, the solution to the boundary value problem, Eq. 6, directly determines the complex impedance.

$$Z/Z_0 \equiv E_{||}(R)/H_{||}(R) = \frac{J_0(\varpi R/c)}{J_1(\varpi R/c)} = \frac{\varpi}{c}\,\Lambda(\varpi)^{-1}\,. \tag{8}$$

In terms of complex conductivity $Z = Z_0\sqrt{\omega/4\pi i\sigma(\omega)}$ and thus, $Z_s/Z_n = \sqrt{\sigma_n(\omega)/\sigma_s(\omega)}$. This result is valid in the limit $c/\omega \gg \lambda_{s,n}$, which is well satisfied for SRF cavities at GHz frequecies. The real and imaginary parts of $Z_{s/n} = R_{s/n} - iX_{s/n}$ are the surface resistance and surface reactance of the metal in the superconducting (s) and normal (n) states, respectively. For SRF cavities in the normal state with a moderate level of disorder we have $\omega\tau \ll 1$, in which case $\sigma_{1n} \simeq \sigma_{\rm D}$ and $\sigma_{2n} \simeq (\omega\tau)\sigma_{\rm D} \ll \sigma_{1n}$. Thus,

$$X_n \simeq R_n \simeq Z_0\sqrt{\omega/8\pi\sigma_{\rm D}}\,. \tag{9}$$

The change in the reactance upon cooling through the superconducting transition generates a shift in the resonance frequency,

$$\delta f \equiv f_s - f_n = \frac{f}{2G}\,(X_n - X_s)\,, \tag{10}$$

while the surface resistance below T_c determines the quality factor,

$$Q_s = \frac{G}{R_s}\,, \tag{11}$$

where G is the geometric factor of the cavity, which for the $\rm TM_{010}$ mode of a cylindrical cavity is given by Eq. (5).

Frequency shift, theory and experiment

Figure (1) shows the remarkable agreement between theory and experiment for the frequency shift of a $f = 2.6\,$GHz cavity relative to the normal state just above T_c.
The theoretical curve corresponds to a quasiparticle scattering time of $\tau = 2.57\times10^{-13}\,$sec ($\tau = 1.963\,\tau_0$). This result is in reasonable agreement with the value of τ obtained from the reported normal-state suface resistance. We estimate τ from R_n using Eq. (9) as

$$\tau = \pi\,f\,\left(\frac{\lambda_{\rm L_0}}{c}\,\frac{Z_0}{R_n}\right)^2\,. \tag{12}$$

Bafia et al. [3] report $R_n \simeq 7.1\times10^{-3}\Omega$, which gives $\tau = 0.28\,ps$, compared to the value of $\tau = 0.26\,ps$ obtained by fitting to the temperature-dependent frequency shift (see also [4]).

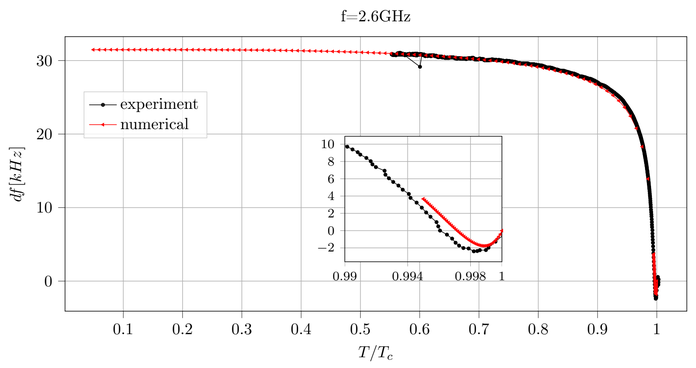

Figure 1: Comparing the theoretical estimate (red line) and experimental measurement (black circles) of $f = 2.6\,GHz$ cavity.

Normal metal skin depth

In the limit $\omega\tau \ll 1$, $\sigma_n \simeq \sigma_{\rm D}$ determines the dissipation and the penetration depth of the EM field in the normal metal. On the metallic side of the interface ($x > 0$),

$$A_n(x,\omega) = A_0\,\Re e^{iq(\omega)x}\,, \tag{13}$$

where $q(\omega) = \sqrt{i\,4\pi\sigma_{\rm D}\omega}/c = (1+i)/\delta(\omega)$ with

$$\delta(\omega) \equiv c/\sqrt{2\pi\sigma_{\rm D}\omega} = \frac{\lambda_{\rm L_0}}{\sqrt{\omega\tau/2}}\,. \tag{14}$$

Note that $\lambda_{\rm L_0} = c/\omega_p$ is the clean limit result for the London penetration depth at zero temperature, defined here in terms of the plasma frequency, $\omega_p \equiv \sqrt{\frac{8\pi}{3}N_f e^2 v_f^2}$. The penetration of the EM field in the normal metal is then given by

$$\lambda_n = \frac{1}{A_0}\int_0^\infty dx\,A_n(x,\omega) = \Re\left(\frac{i}{q(\omega)}\right) = \frac{\delta(\omega)}{2}\,. \tag{15}$$

Anomalous frequency shift

Here we provide an approximate analytic expression for the response function and the frequency shift anomaly near T_c. The real part of the current response function in Eq. (3) is proportional to the out of phase component of conductivity, and thus the penetration depth in the superconducting state,

$$\Re K(\omega) = \frac{4\pi\omega}{c^2}\,\sigma_2(\omega)\xrightarrow[T\to T_c^-]{\hbar\omega\ll 2\Delta(T)}\frac{1}{\lambda_{\rm L}(T,\tau)^2}\,, \tag{16}$$

where $\lambda_{\rm L}(T,\tau) \simeq \lambda_{\rm L}/\sqrt{1-T/T_c}$ for $T \lesssim T_c$ for $T \simeq T_c^-$. The imaginary part of $K(\omega)$ is proportional to the dissipative component of the conductivity,

$$\Im K(\omega) = -\frac{4\pi\omega}{c^2}\,\sigma_1(\omega)\xrightarrow[T\to T_c^-]{\hbar\omega\ll 2\Delta(T)} -\frac{1}{2\lambda_{\rm n}(\omega,\tau)^2}\,, \tag{17}$$

with $\lambda_n(\omega,\tau)$ given by Eqs. (15) and (14), i.e. we drop corrections of order $(\Delta(T)/\pi T_c)^2$ to the normal-state Drude conductivity.
Thus, for temperatures just below the superconducting transition, but still below the pair-breaking continuum, $\hbar\omega \ll 2\Delta(T)$, we obtain an approximate analytic form for the frequency shift and quality factor using Eq. (6) in the limit $\delta\omega/\omega \ll 1$ and $1/Q \ll 1$. In particular, the frequency shift of the superconductor *relative to the ideal cavity frequency* for $|T - T_c| \ll T_c$ becomes,

$$\delta f_s = -\frac{f}{R_{\rm eff}}\,\Re\left\{\left(\frac{1}{\lambda_{\rm L}(T,\tau)^2} - \frac{i}{2\lambda_n(\omega,\tau)^2}\right)^{-\frac{1}{2}}\right\}\,, \tag{18}$$

where $R_{\rm eff} = R\,G_{\rm Tesla}/G_{\rm cyl}$ and R is the radius of a cylindrical SRF cavity with a $\rm TM_{010}$ mode frequency of $f = 2.6\,$GHz. The corresponding shift for the normal metallic state just above T_c is $\delta f_n = -(f/R_{\rm eff})\,\lambda_n$. Thus, the observable frequency shift of the superconducting cavity relative to the normal state is $\delta f = \delta f_s - \delta f_n$. This approximate result reproduces the negative frequency shift near T_c, including an good estimate of the temperature, and minimum value, of the negative frequency shift, but deviates from the full theory as the temperature drops further. Thus, the origin of the negative shift is the competition between the large normal-state penetration depth for disordered Nb and the London penetration depth which competes with λ_n only as $T \to T_c^-$.

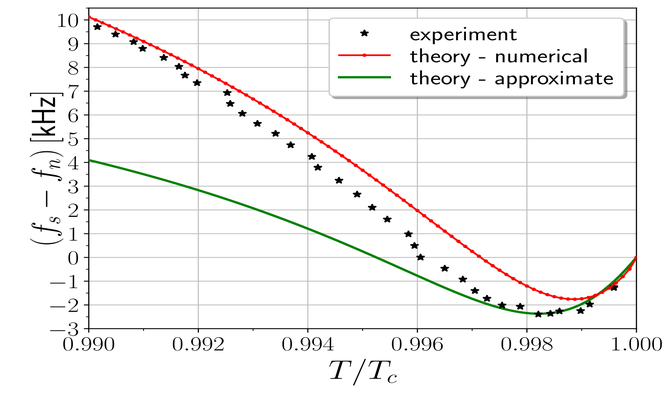

Figure 2: The experimental data for the frequency shift of the 2.6 GHz cavity near T_c are the black symbols. The minumum frequency shift is $-2.39\,$kHz at $T = 0.9982\,T_c$, and extends over a temperature window of $|\Delta T| \approx 4\times10^{-3}\,T_c$. The theoretical result based on the best fit to the full temperature dependence of the frequency shift is the red line. The green line is the approximate theoretical result based on the leading terms in the current response function as a function of $t = 1 - T/T_c$ with the constraint $\hbar\omega \ll 2\Delta(T)$ for the same scattering time, $\tau = 0.257\,ps$.

Quality factor

Figure 3 is shows the dependence of Q on the quasiparticle scattering time in units of $\tau_\Delta \equiv \hbar/2\pi\Delta(T)$ for an $f = 2.6GHz$ Nb SRF cavity at $T = 0.2T_c$. Thus, in the local limit, i.e. sufficient disorder such that $\lambda_{\rm L} \gg \xi$, the maximum Q occurs for intermediate disorder with $\tau \sim \tau_\Delta$. Indeed the non-monotonic dependence of Q on τ is due to the pair-breaking suppression of the superfluid fraction, $n_s \propto 1/\lambda^2$, by quasiparticle-impurity scattering. For strong disorder, $\tau \ll \tau_\Delta$, the quality factor drops rapidly and approaches the Q of the normal state for $\tau \to 0$. In the clean limit $\tau \to \infty$ the local approximation for the current response function breaks down for Nb.

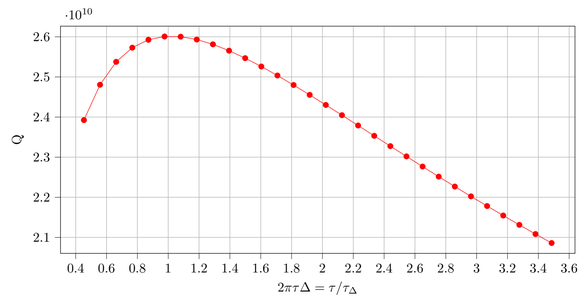

Figure 3: Theoretical result for the quality factor Q_s at $T/T_c = 0.2$ as a function of the quasiparticle scattering time in units of τ_Δ, for a $f = 2.6GHz$ cavity.

Conclusion

The resonant frequency shift and quality factor of disordered Nb SRF cavities is derived by solving the boundary conditions on EM wave at the vacuum-metal interface. We present an analytic approximation to the negative frequency shift, its temperautre width and depth.
The pair-breaking effect of disorder leads to the non-monotonic dependence of the quality factor on the quasiparticle scattering rate.
Theoretical results for the frequency shift and quality factor, as well as the suppression of T_c and superfluid density by disorder, can provide powerful analysis tools for characterizing disorder in SRF cavities for both accelerator and detector applications.

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