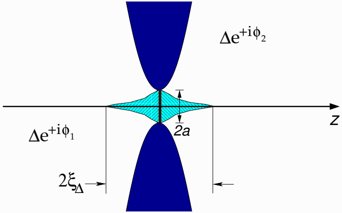
Quantum Processes in Josephson Junctions & Weak Links

J. A. Sauls

Northwestern University



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- Erhai Zhao, George Mason University   Tomas Löfwander, Chalmers University

Preface on Dirac Materials

Dirac materials

- Materials whose low energy electronic properties are a direct consequence of Dirac spectrum E = vk
- How do we “design” Dirac Materials?
- Can be a collective state: 3He superfluid, heavy fermion, organic, high Tc superconductors
- Band structure effect – graphene, Topological states

T. Wehling, A Black-Schaffer and A. V. Balatsky,
Dirac Materials, Adv Phys 2014

Dirac Fermions & Zero Energy Bound States

- Dirac Fermion coupled to a Scalar Bose Field

$$i\hbar\partial_t|\psi\rangle = (-i\hbar c\vec{\alpha}\cdot\nabla + \beta g\Phi)|\psi\rangle$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad |\psi\rangle = \text{col}(\psi_1, \psi_2, \psi_3, \psi_4)$$

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- Broken Symmetry State: $\Phi = \Phi_0 \rightsquigarrow$ **Mass**: $Mc^2 = g\Phi_0 \rightsquigarrow E_\pm = \pm\sqrt{c^2|\mathbf{p}|^2 + (Mc^2)^2}$

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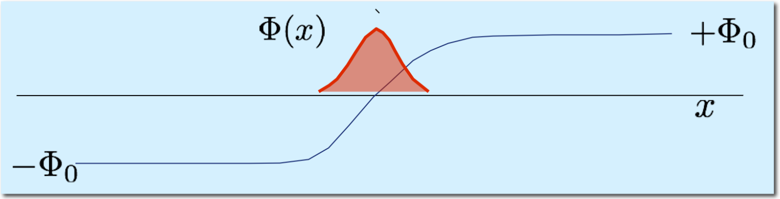
- Degenerate Vacuum States: $\Phi(x \to \pm\infty) = \mp\Phi_0$:
- "Zero Mode" $\rightsquigarrow$ Fermion with $E = 0$ confined on the the Domain Wall :



"Topologically Protected" Zero Mode

R. Jackiw and C. Rebbi, Phys. Rev. D 1976

Nambu-Dirac Fermions in Superconductors

- Bogoliubov-Nambu Equations - *particle-hole coherence*:

$$\begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 - \mu & 0 \\ 0 & \frac{\hbar^2}{2m}\nabla^2 + \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & \Delta \\ \Delta^\dagger & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}$$

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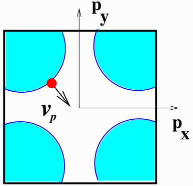
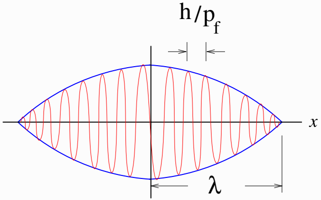
- Separation of scales: $\hbar/p_f \ll \hbar v_f/\Delta \leq \lambda$: $\rightsquigarrow u = U_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$





- Nambu-Dirac Spinors coupled to the (Bosonic) Cooper-Pair Field

$$\hbar\mathbf{v}_{\mathbf{p}}\cdot\nabla_{\mathbf{r}} \begin{pmatrix} U \\ -V \end{pmatrix} + \begin{pmatrix} 0 & \Delta(\mathbf{p},\mathbf{r}) \\ \Delta^\dagger(\mathbf{p},\mathbf{r}) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \varepsilon \begin{pmatrix} U \\ V \end{pmatrix}$$

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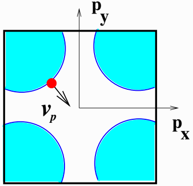

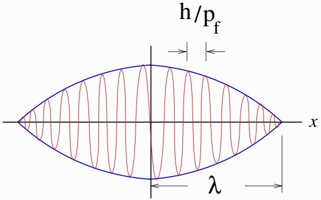

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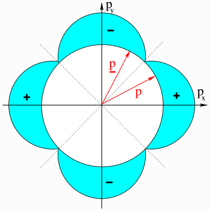
- Zero Modes if $\Delta(x = -\infty) = -\Delta(x = +\infty)$ along $x = \hat{\mathbf{v}}_{\mathbf{p}} \cdot \mathbf{r}$

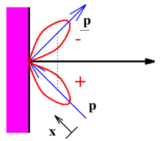
- Andreev's Equation for Coherent Electron-Hole States

$$\hbar \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \begin{pmatrix} U \\ -V \end{pmatrix} + \begin{pmatrix} 0 & \Delta(\mathbf{p},\mathbf{r}) \\ \Delta^{\dagger}(\mathbf{p},\mathbf{r}) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \varepsilon \begin{pmatrix} U \\ V \end{pmatrix}$$

- $\Delta(\mathbf{p}) = \Delta(\hat{\mathbf{p}}_x^2 - \hat{\mathbf{p}}_y^2)$



$[110]$ reflection:



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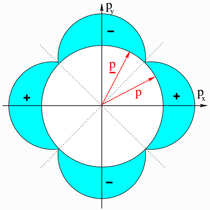

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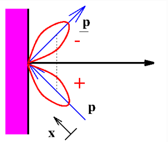
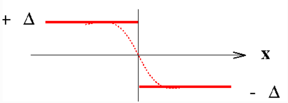
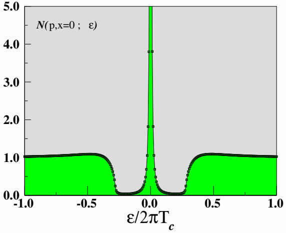


- 

- Electron & Hole Bound State:

$$|\psi\rangle \sim \sqrt{|\Delta(\mathbf{p})|} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad e^{-2|\Delta(\mathbf{p})|\,|x|/\hbar v_f}$$



- *Tunneling into Surface States of HTC Superconductors*, PRL 79:281–284 (1997), M. Fogelström, D. Rainer, & J. A. Sauls

Josephson Tunneling in Superconductors

- B. Josephson, Phys. Lett. 1, 251 (1962).
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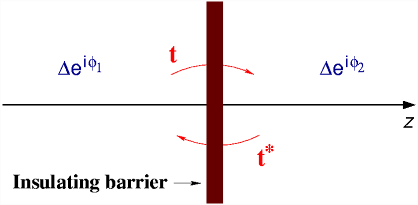
$$H = H_1 + H_2 + H_{tH}$$



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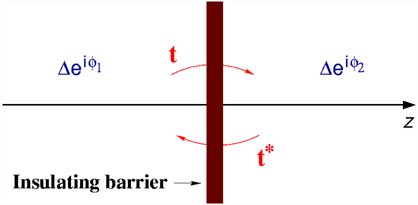

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$$H_1 = \sum_{k\sigma} \xi_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma} + \frac{1}{2} \sum_{k\sigma} \left(\Delta_k c^\dagger_{k\sigma} c^\dagger_{-k-\sigma} + \Delta^*_k c^\dagger_{-k-\sigma} c_{k\sigma} \right)$$

$$H_2 = \sum_{p\sigma} \xi_{p\sigma} a^\dagger_{p\sigma} a_{p\sigma} + \frac{1}{2} \sum_{p\sigma} \left(\Delta_p a^\dagger_{p\sigma} a^\dagger_{-p-\sigma} + \Delta^*_p a^\dagger_{-p-\sigma} a_{p\sigma} \right)$$

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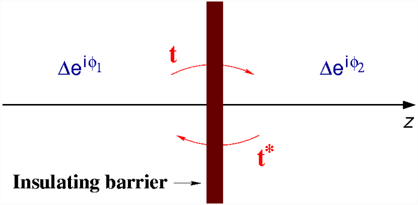

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- $\langle I \rangle = e \langle \dot{N}_2(t) \rangle = 2e\, \mathrm{Im} \sum_{p,k,\sigma} t_{p,k} \langle a^\dagger_{p\sigma}(t) c_{k\sigma}(t) \rangle$

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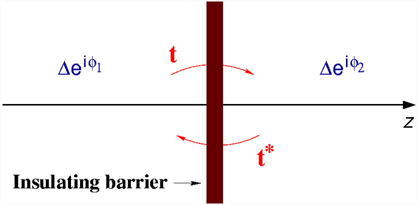

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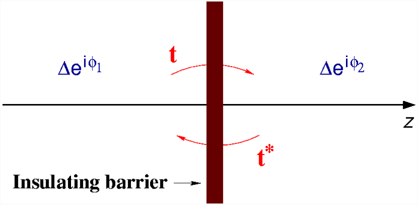

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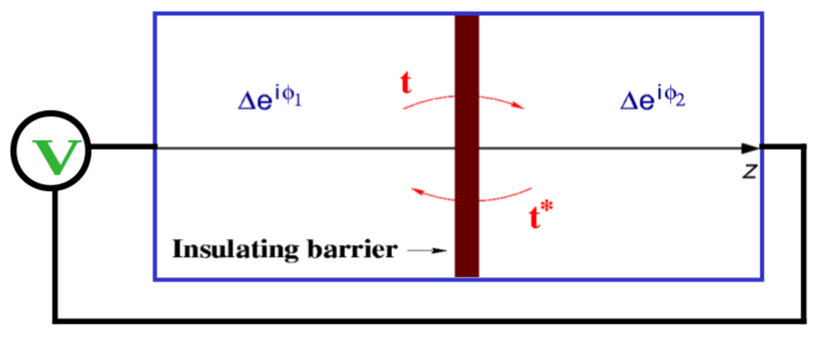
a.c. Josephson Effects



- Supercurrent: $I_s = I_c(T)\sin(\phi_t)$

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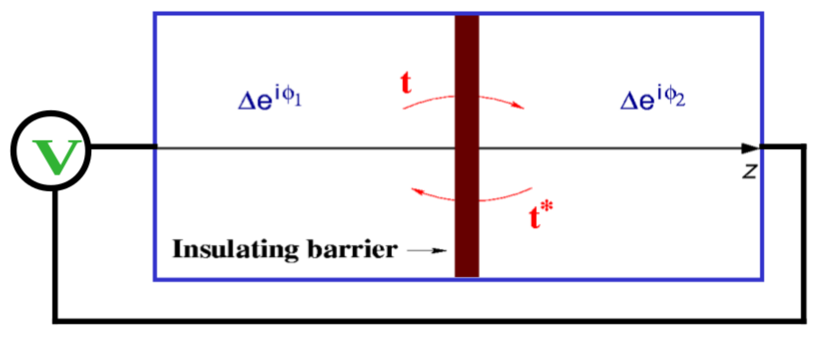

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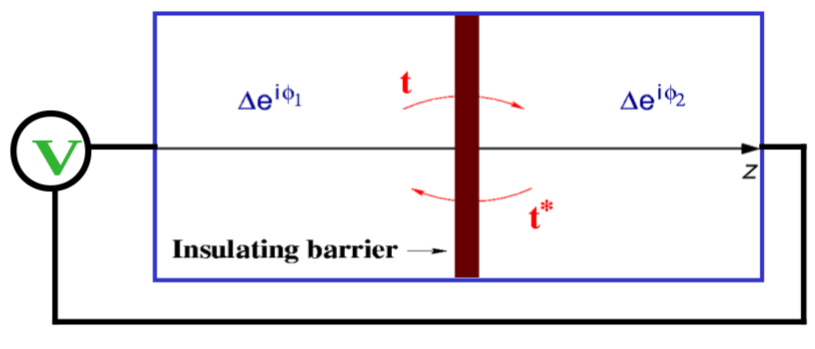

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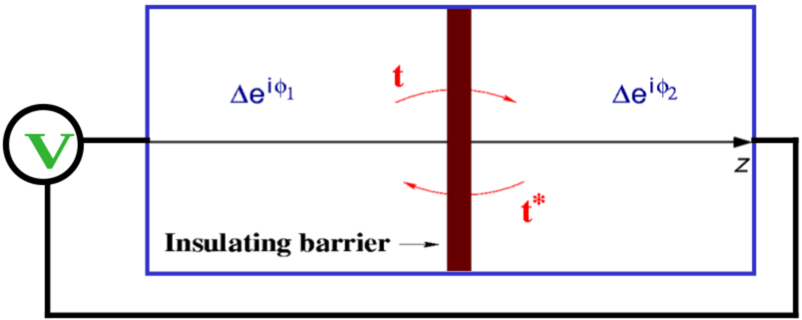

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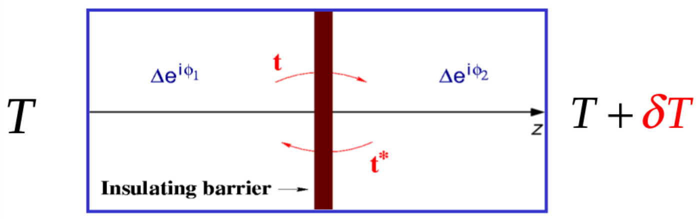
    - What is the origin of phase-dependent dissipation?

Linear Response to a Thermal Bias

- Maki & Griffin, PRL (1965); Guttman et al. PRB 57, 2717 (1998)



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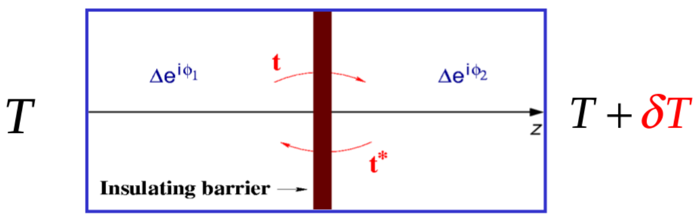

Heat Current: Tunneling Hamiltonian

- $I_Q = -i \sum_{p,k,\sigma} \left\{ t_{p,k} \left(\xi_{p\sigma}\, a^{\dagger}_{p\sigma} c_{k\sigma} - \Delta_p\, a^{\dagger}_{p\sigma} c_{-k-\sigma} \right) - h.c. \right\}$

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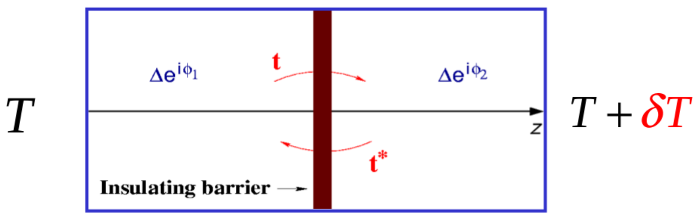

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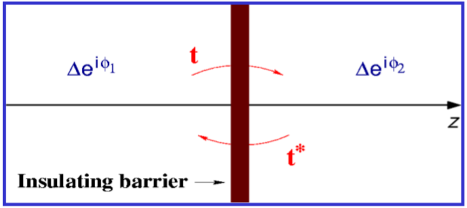
T



$T + \delta T$

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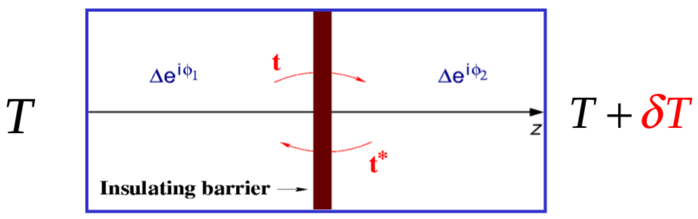

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$\rightsquigarrow$ **Failure of Linear Response Theory?**

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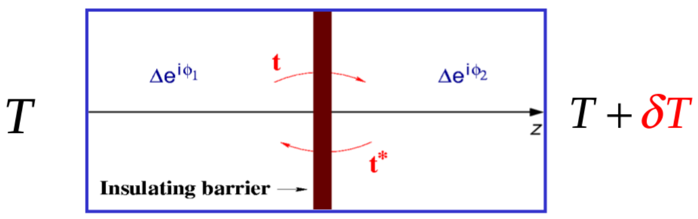

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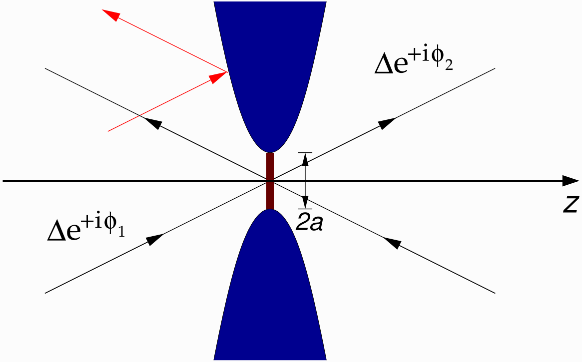
Non-Perturbative Theory of Transport in Phase-Biased Josephson Junctions

- Phase Bias: $\phi = \phi_2 - \phi_1$
- Thermal Bias: $\delta T = T_2 - T_1$
- Barrier Transmission: $0 < D \leq 1$
- Mesoscopic Junction: $\hbar/p_f \ll a < \xi_\Delta$



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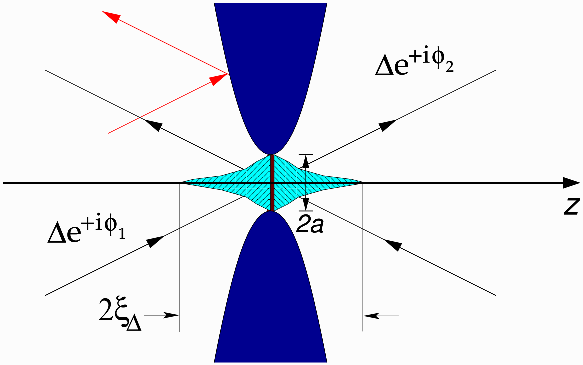

Josephson Phase $\rightsquigarrow$ *New* Electronic States Confined to the Interface !

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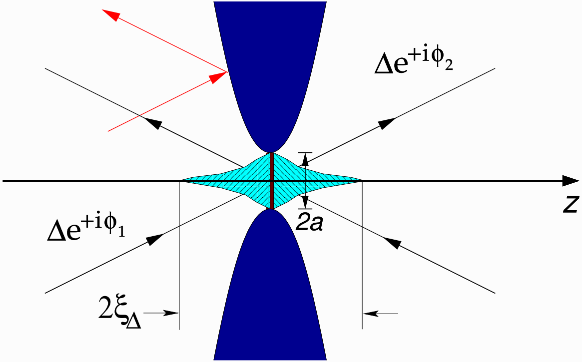
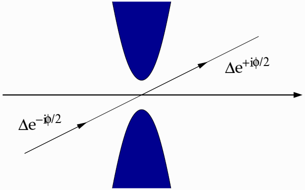


Josephson Phase $\rightsquigarrow$ *New* Electronic States Confined to the Interface !

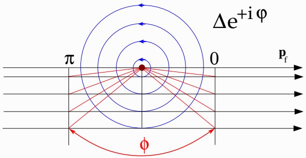
- Energy & Phase-dependent Transmission: $D \rightsquigarrow \mathscr{D}(\varepsilon, \phi)$

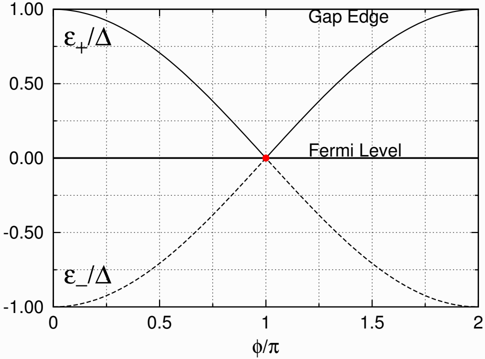
Fermion Bound States of a Josephson Weak Link or 2π Vortex



Sharvin Contact

$$\varepsilon_{\pm}(\phi) = \pm|\Delta|\,|\cos(\phi/2)|$$





- Andreev's Equation for Coherent Electron-Hole States

$$\hbar \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \begin{pmatrix} U \\ -V \end{pmatrix} + \begin{pmatrix} 0 & \Delta(\mathbf{p},\mathbf{r}) \\ \Delta^{\dagger}(\mathbf{p},\mathbf{r}) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \varepsilon \begin{pmatrix} U \\ V \end{pmatrix}$$

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$$\gamma(\mathbf{p},\mathbf{r};\varepsilon) = U/V \qquad \bar{\gamma}(\mathbf{p},\mathbf{r};\varepsilon) = V/U$$

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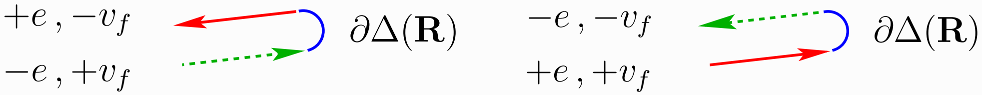
- Nonequilibrium: $\gamma(\mathbf{p},\mathbf{r};\varepsilon,t)$



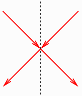
Amplitude: $\gamma(\mathbf{p},\mathbf{r};\varepsilon,t)$ Amplitude: $\bar{\gamma}(\mathbf{p},\mathbf{r};\varepsilon,t)$

"h-e and e-h branch conversion scattering"

Multiple Scattering from a Potential + Branch conversion scattering

Multiple Scattering from a Potential + Branch conversion scattering

- Interface Potential Scattering:
  S-matrix



$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

Multiple Scattering from a Potential + Branch conversion scattering

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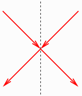

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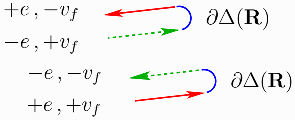
- Andreev Scattering: Riccati



$$\gamma(\mathbf{p},\mathbf{r};\varepsilon,t) \qquad \bar{\gamma}(\mathbf{p},\mathbf{r};\varepsilon,t)$$

Multiple Scattering from a Potential + Branch conversion scattering

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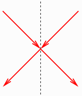

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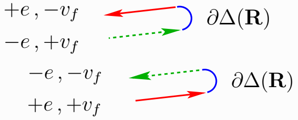

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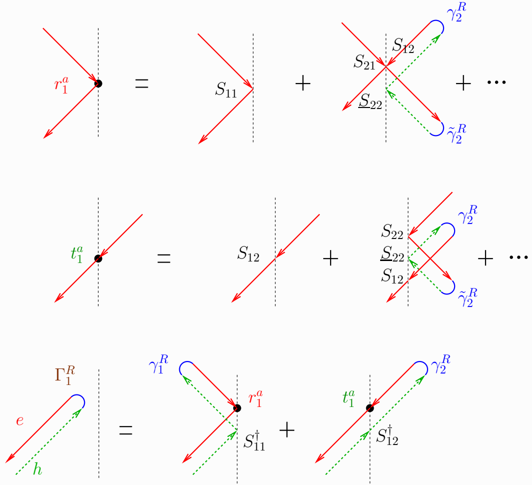

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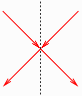

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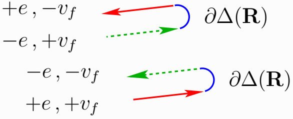

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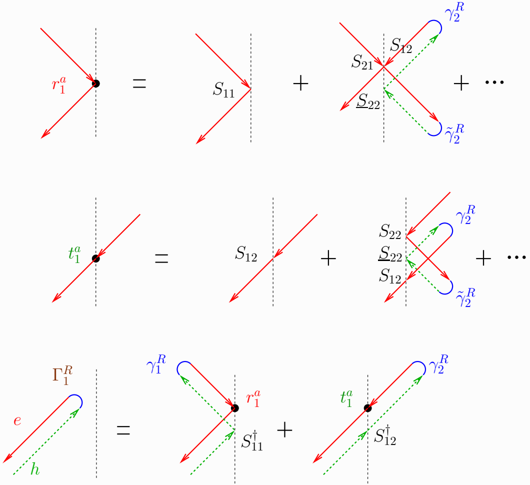
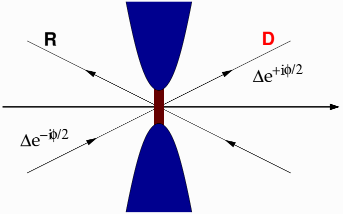


- Bound-States $\rightsquigarrow$ Poles of the re-normalized S-matrix amplitudes

- *Nonequilibrium Superconductivity Near Spin-Active Interfaces*, PRB 70:134510 (2004), E. Zhao, T. Löfwander & J. A. Sauls

Fermion Bound States of a Josephson Junction



$$S = \begin{pmatrix} \sqrt{R} & i\sqrt{D} \\ i\sqrt{D} & \sqrt{R} \end{pmatrix}$$

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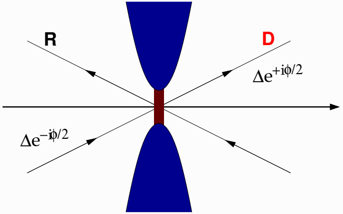
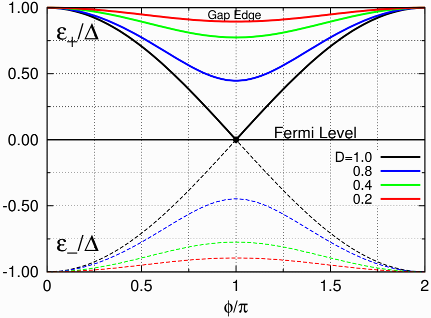


$$S = \begin{pmatrix} \sqrt{R} & i\sqrt{D} \\ i\sqrt{D} & \sqrt{R} \end{pmatrix}$$

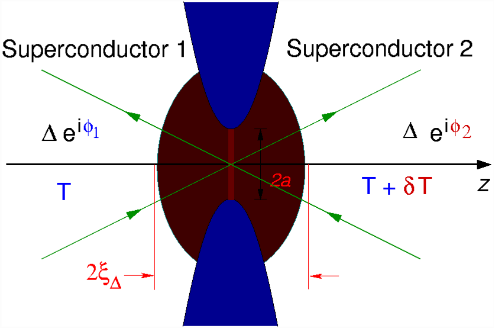
$$\varepsilon_{\pm}(R,\phi) = \pm\Delta\sqrt{\cos^2(\phi/2) + R\sin^2(\phi/2)}$$

$$0 < D \leq 1$$

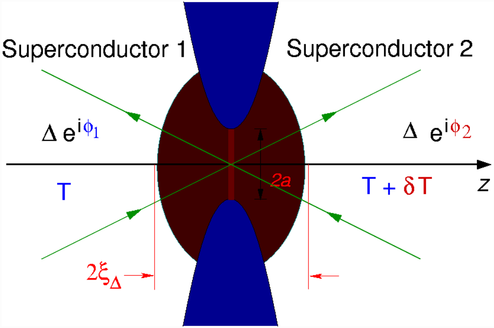
Potential + Andreev Scattering $\rightsquigarrow$ *Gap in the Bound-State Dispersion*



Heat Transport through a Phase-Biased Josephson Junction



- Heat Current $j_Q = -\kappa\, \delta T$
- Carriers = *bulk* quasiparticles



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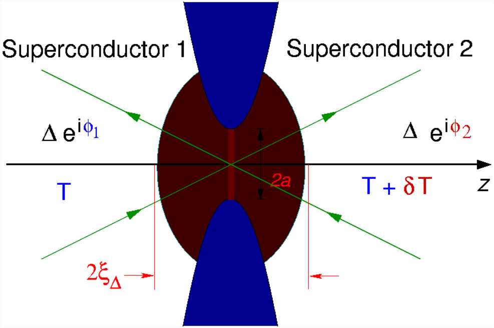
- $N_{\text{bulk}}(\varepsilon) = N(0)\, \dfrac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}}$



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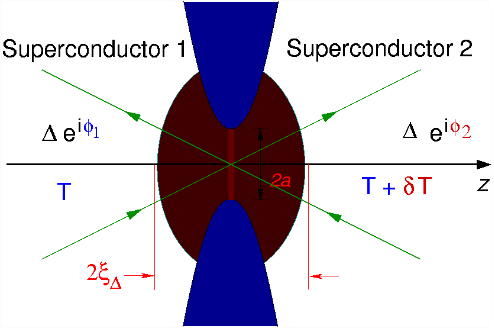

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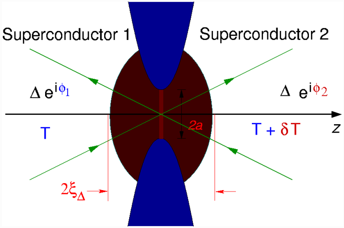
Thermal Conductance

$$\kappa(\phi,T) = A\int_{\Delta}^{\infty} d\varepsilon\, N_{\text{bulk}}(\varepsilon)\,[\varepsilon\, v_{\text{g}}(\varepsilon)]\,\mathscr{D}(\varepsilon,\phi)\left(-\frac{\partial f}{\partial T}\right)$$

- $\mathscr{D}(\varepsilon,\phi)$ = Quasiparticle Transmission Probability

Transmission Probability for a Phase-Biased Josephson Junction

$$\mathscr{D}(\varepsilon,\phi) = \mathscr{D}_{e\to e}(\varepsilon,\phi) = \mathscr{D}_{e\to h}(\varepsilon,\phi)$$



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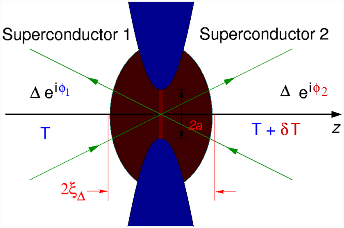

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Excitations: $\varepsilon \geq \Delta$

- Direct Transmission:

$$\mathscr{D}_{e\to e}(\varepsilon,\phi) = D\,\frac{(\varepsilon^2-\Delta^2)\,(\varepsilon^2-\Delta^2\cos^2(\phi/2))}{[\varepsilon^2-\Delta^2\,(1-D\sin^2(\phi/2))]^2}$$

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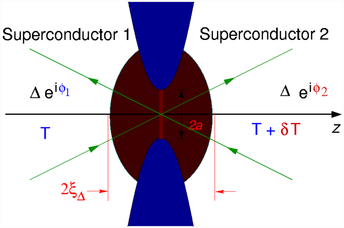

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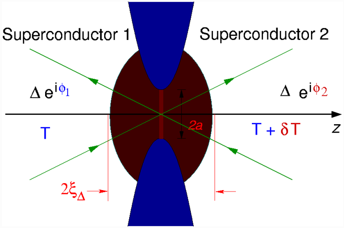

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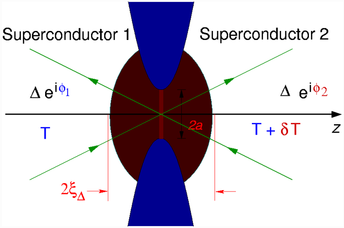

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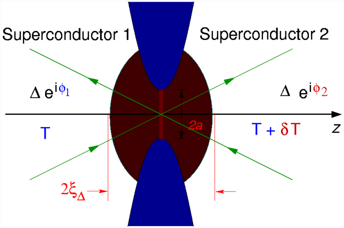

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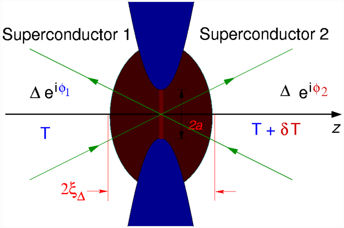
- $D \ll 1$ $\quad \mathscr{D}(\varepsilon,\phi) = D\dfrac{(\varepsilon^2-\Delta^2\cos^2(\phi/2))}{\varepsilon^2-\Delta^2} \rightsquigarrow$ Tunneling Limit

Transmission Probability for a Phase-Biased Josephson Junction



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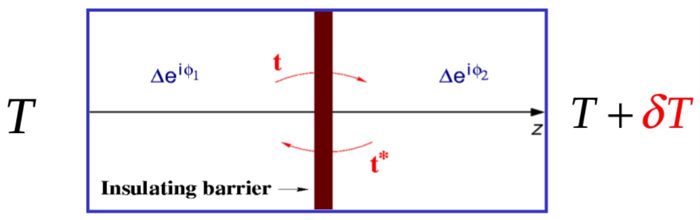
Essential Singularity

Linear Response to a Thermal Bias

- Maki & Griffin, PRL (1965); Guttman et al. PRB 57, 2717 (1998)



Heat Current: Tunneling Hamiltonian

- $I_Q = -i \sum_{p,k,\sigma} \left\{ t_{p,k} \left(\xi_{p\sigma} a^{\dagger}_{p\sigma} c_{k\sigma} - \Delta_p a^{\dagger}_{p\sigma} c_{-k-\sigma} \right) - h.c. \right\}$
- $\langle I_Q \rangle = \delta T \times \underbrace{4\pi N(0) \langle |t|^2 \rangle_{FS}}_{\propto D_{tH}} \underbrace{\int_{\Delta}^{\infty} d\varepsilon \left(-\frac{\partial f}{\partial T} \right)}_{\text{thermal excitations}} \underbrace{\left(\frac{\varepsilon^2 - \Delta^2 \cos(\phi)}{\varepsilon^2 - \Delta^2} \right)}_{\text{Trouble!!}} \to \infty$

$\rightsquigarrow$ **Failure of Linear Response Theory?**

Andreev's Demon & Resonant Transmission

Direct Transmission

$$\mathscr{D}_{e\to e}(\varepsilon,\phi) = D\,\frac{(\varepsilon^2-\Delta^2)(\varepsilon^2-\Delta^2\cos^2(\phi/2))}{[\varepsilon^2-\Delta^2\,(1-D\sin^2(\phi/2)]^2}$$

Branch Conversion

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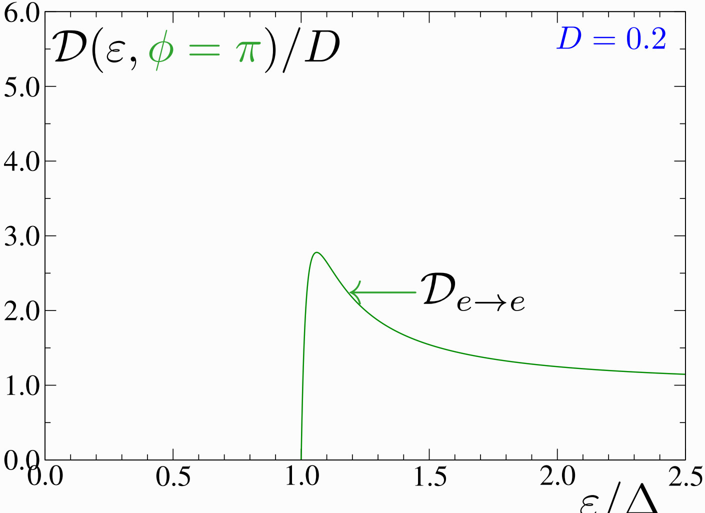

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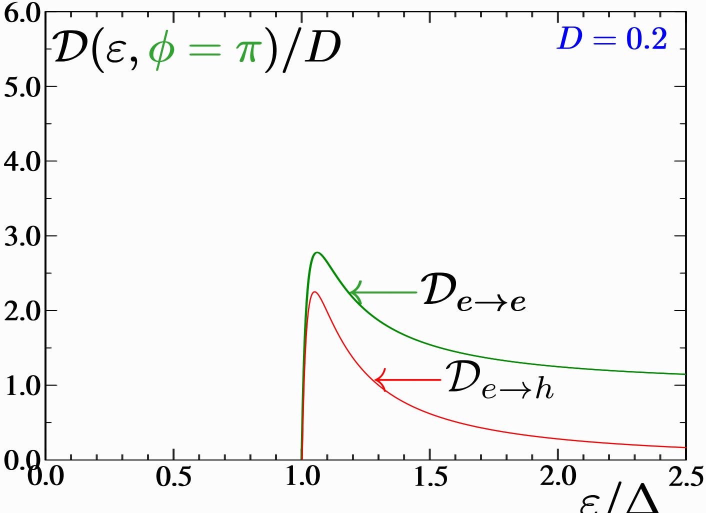

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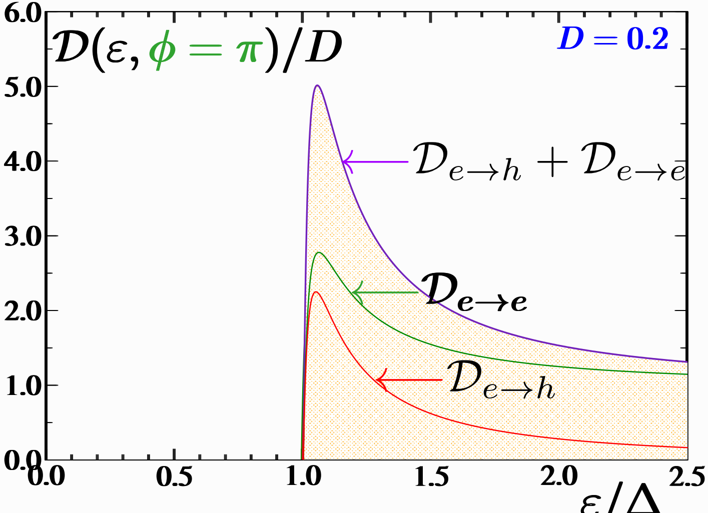
Branch Conversion

$$\mathscr{D}_{e\to h}(\varepsilon,\phi) = DR\,\frac{(\varepsilon^2-\Delta^2)\Delta^2\sin^2(\phi/2)}{[\varepsilon^2-\Delta^2\,(1-D\sin^2(\phi/2)]^2}$$



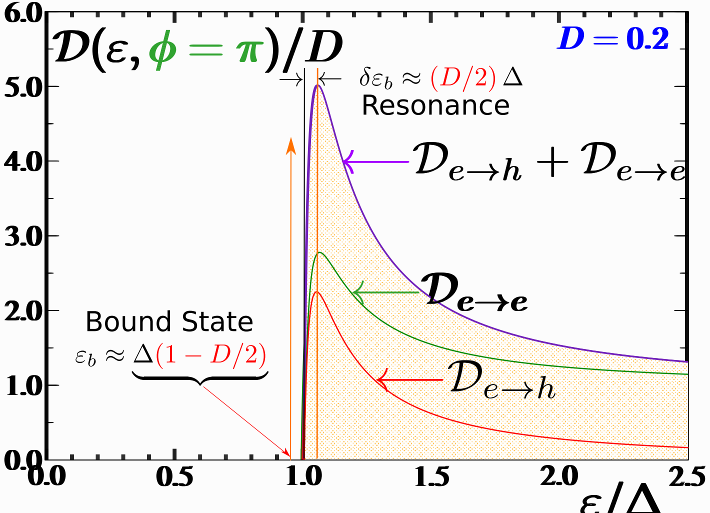
Andreev's Demon & Resonant Transmission

Direct Transmission

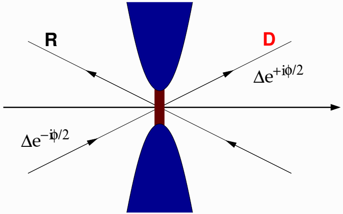
$$\mathscr{D}_{e\to e}(\varepsilon,\phi) = D\,\frac{(\varepsilon^2-\Delta^2)(\varepsilon^2-\Delta^2\cos^2(\phi/2))}{[\varepsilon^2-\Delta^2\,(1-D\sin^2(\phi/2)]^2}$$

Branch Conversion

$$\mathscr{D}_{e\to h}(\varepsilon,\phi) = DR\,\frac{(\varepsilon^2-\Delta^2)\Delta^2\sin^2(\phi/2)}{[\varepsilon^2-\Delta^2\,(1-D\sin^2(\phi/2)]^2}$$

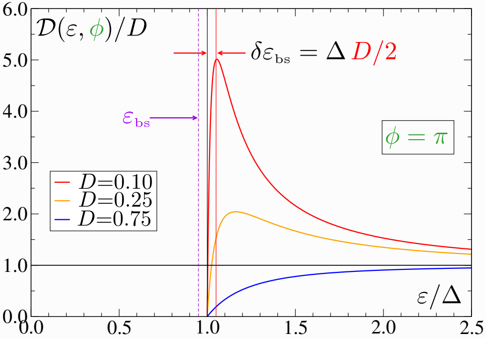


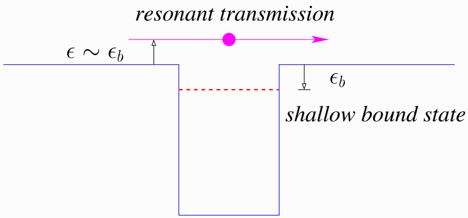
Transmission Resonance for Heat Transport



$$I_\varepsilon(\phi,T) = -\,\kappa(\phi,T)\,\delta T, \quad \text{with} \quad \delta T = T_2 - T_1$$

$$\kappa(\phi,T) = 4A\int_\Delta^\infty d\varepsilon\,\mathscr{N}(\varepsilon)\left[\varepsilon v_g(\varepsilon)\right]\,\mathscr{D}(\varepsilon,\phi)\,\frac{\partial f}{\partial T}$$



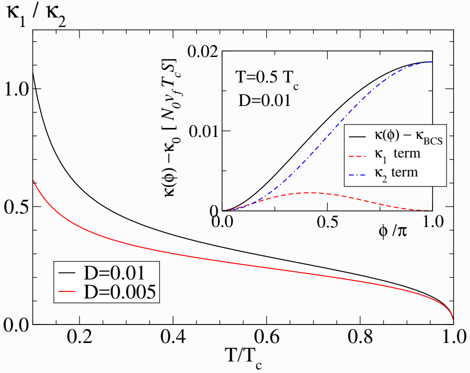


Non-analyticity of the Thermal Conductance

- Tunneling Hamiltonian: $\kappa^{\text{tH}} = \kappa^{\text{tH}}_{\text{BCS}} + \kappa^{\text{tH}}_{2} \sin^2(\phi/2)$ ... But $\kappa^{\text{tH}}_{2} \to \infty$
- Self-Consistent S-matrix for $D \ll 1$:

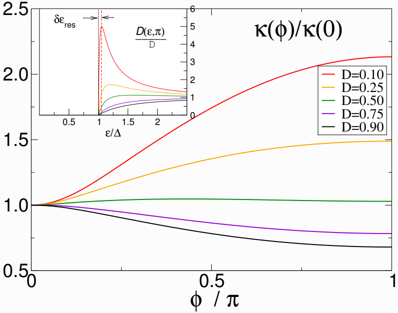
$$\kappa = \kappa_{\text{BCS}} - \kappa_1 \sin^2(\phi/2) \ln\left(\sin^2(\phi/2)\right) + \kappa_2 \sin^2(\phi/2)$$

- $\kappa_{1,2} \xrightarrow[D\to 0]{} D \ln D \Rightarrow$ Finite, but Non-Analytic and Non-perturbative

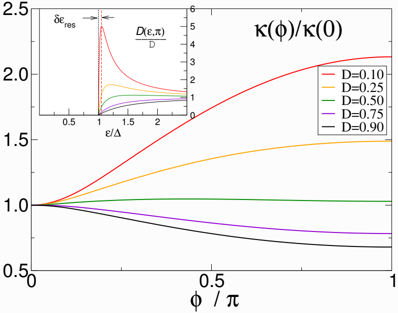


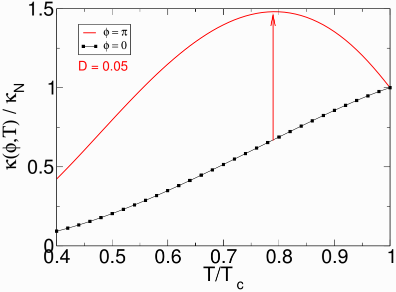
- Andreev Bound-State Formation is *non-perturbativce*

Phase-Tuneable Resonant Enhancement of the Heat Current



Phase-Tuneable Resonant Enhancement of the Heat Current

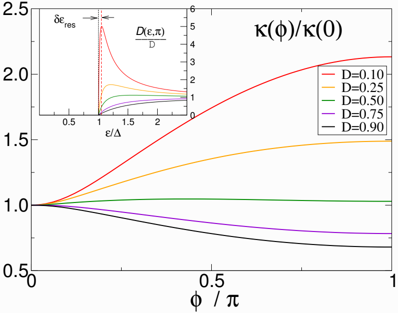


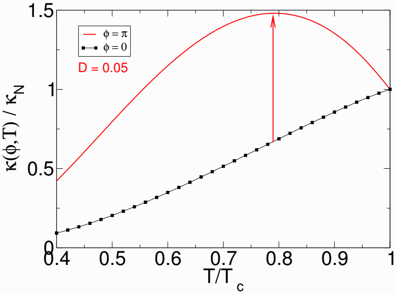


Andreev's Demon $\rightsquigarrow$ Fermion Bound States "control" thermal transport

- $\phi = 0$: $\kappa \downarrow$ for $T < T_c$.
- $\phi = \pi$: $D < 0.5$ $\kappa(T) \uparrow$ below T_c.
- $D \gtrsim 0.5$: $\kappa(\phi) < \kappa(0)$

Phase-Tuneable Resonant Enhancement of the Heat Current





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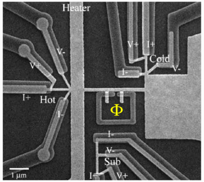
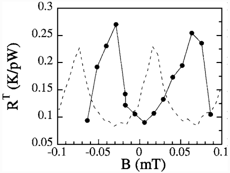
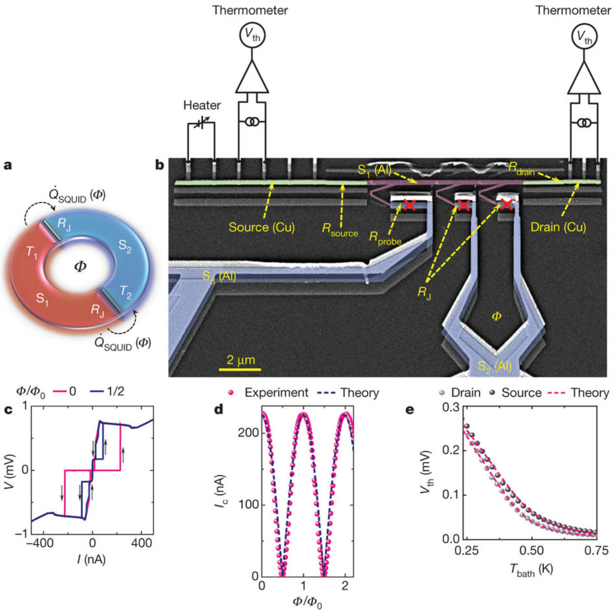




Northwestern LT Group: Z. Jiang et al., PRB 72, 020502 (2005)

The Josephson heat interferometer - Giazotto et al. Nature 492, 401 (2012)



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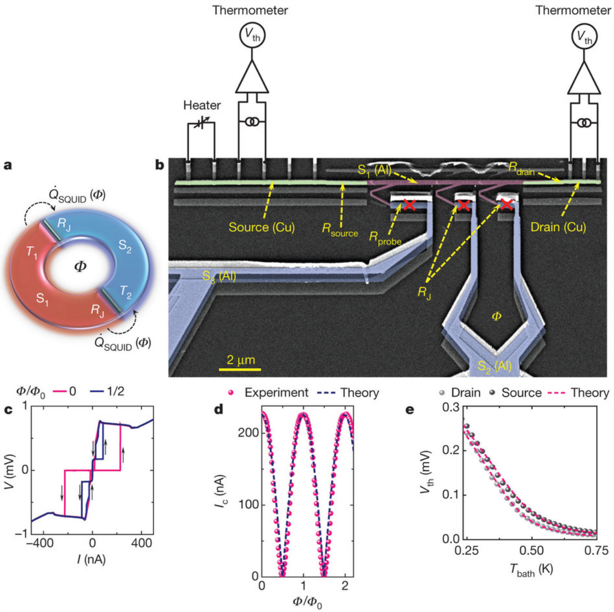
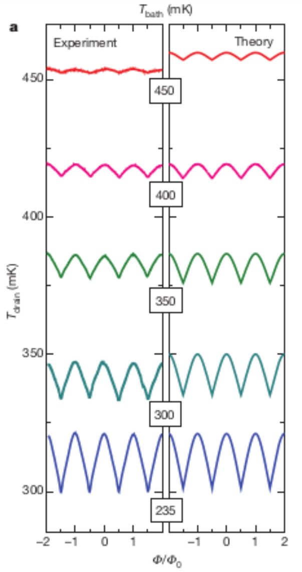
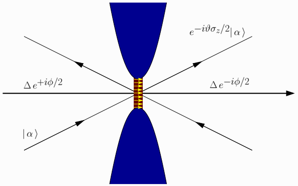


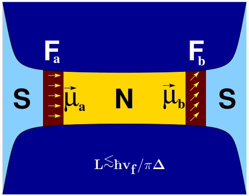


Superconducting-Ferromagnetic-Superconducting Junctions



Magnetic control of Charge

- Tuneable superconducting transition
- Spin-triplet pairing correlations
- π junctions



Voltage control of Spin

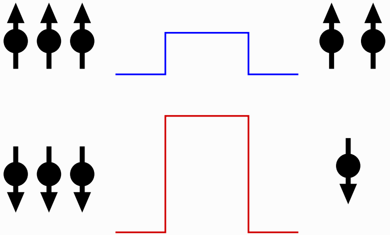
- Spin valves - Spin supercurrents
- SC Spin-Transfer Torque
- Spin manipulation

Nonequilibrium quantum transport theory in S/F heterostructures

Spin-Polarized Supercurrents for Spintronics, Physics Today, Jan. 2011, M. Eschrig

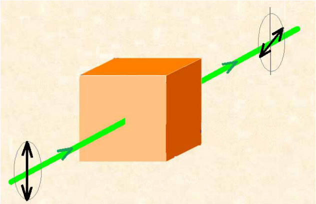
Ferromagnetic Point Contacts

Spin Filtering



Spin-dependent Transmission: $D_\uparrow \neq D_\downarrow$

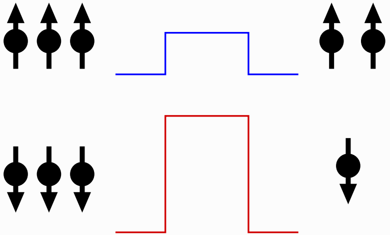
Spin Mixing



Spin Faraday Rotation Angle: ϑ

Ferromagnetic Point Contacts

Spin Filtering



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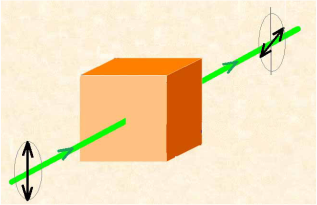
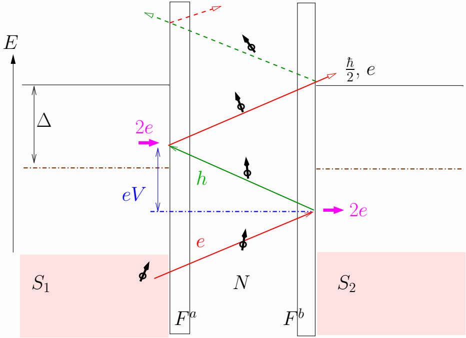


Spin Faraday Rotation Angle: ϑ

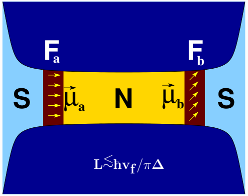
- $S_{11} = S_{22} = \begin{pmatrix} \sqrt{R_\uparrow} e^{i\vartheta/2} & 0 \\ 0 & \sqrt{R_\downarrow} e^{-i\vartheta/2} \end{pmatrix}$
- $S_{12} = S_{21} = i \begin{pmatrix} \sqrt{D_\uparrow} e^{i\vartheta/2} & 0 \\ 0 & \sqrt{D_\downarrow} e^{-i\vartheta/2} \end{pmatrix}$
- $R_{\uparrow/\downarrow} = 1 - D_{\uparrow/\downarrow}$
- $|\uparrow\rangle \to \cos(\vartheta/2)|\uparrow\rangle + \sin(\vartheta/2)|\downarrow\rangle$
- First principles theory of magnetically active interfaces $\rightsquigarrow S(D, \vartheta, \ldots)$
  - Nonequilibrium Superconductivity near Spin-Active Interfaces, PRB 70, 134510 (2004), Zhao, Löfwander, JAS

Quantum Transport Theory of Spin and Charge in SFS Junctions:
spin filtering + spin mixing + MAR

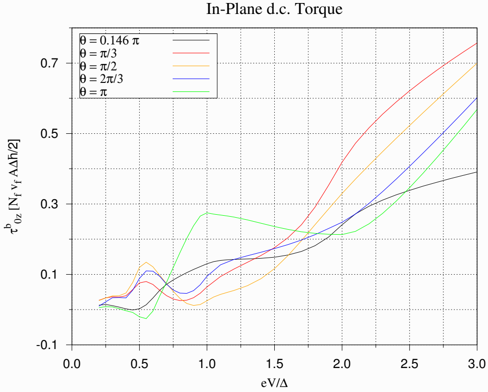


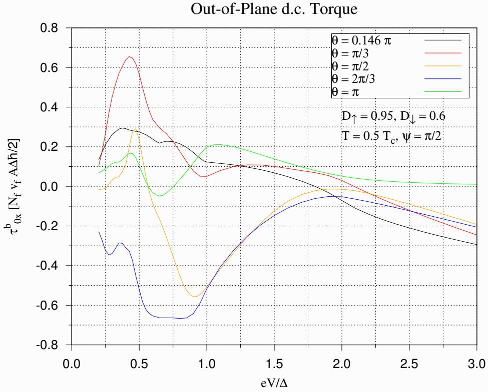
- e/h's scatter inelastically: $\varepsilon \mapsto \varepsilon + m\omega_J$ (mth order MAR).
- e/h's can escape into leads for $\varepsilon > \Delta$
- mth order MAR: transports charge $= m \times 2e$, spin $\hbar/2$

Long-range spin-transfer torque in SFNFS contacts ($L \sim 0.1 - 1.0\mu m$)



- Spin-Transfer Torques:
  $$\vec{\tau}^b(t) = \tau_0^b + \sum_{k=1}^{\infty} \left[\vec{\tau}^b_{k,c} \cos(k\omega_J t) + \vec{\tau}^b_{k,s} \sin(k\omega_J t) \right]$$
- $\tau^b_{0z} \propto N_f (v_f \hbar) V$, $eV \gg \Delta$
- MAR + spin-mixing + $\vec{\mu}_a \times \vec{\mu}_b = \sin(\psi)\hat{\mathbf{x}} \rightsquigarrow \tau^b_{0x}$





Directions and Challenges

- Nano-scale SFS JJs with CNT and Single Molecular Magnets
- Circuit QED with Spin-Triplet Superconductors (Sr_2RuO_4, UPt_3, ?)
- Interacting Classical or Quantum Magnets mediated via Long-Range Josephson Spin-Transfer Torques
- Arrays of SFNFS JJs for Voltage-Controlled Spin Transport ($L \sim 0.1 - 1.0 \mu$m)