International Workshop on Electrons & Ions in Quantum Fluids & Solids, Mishima, Japan, March 11-14, 2018

The Mobility of Electrons in ^{3}He at Very Low Temperatures
The Left Hand of the Electron in Superfluid ^{3}He-A

J. A. Sauls

Northwestern University

- Oleksii Shevtsov (Northwestern)

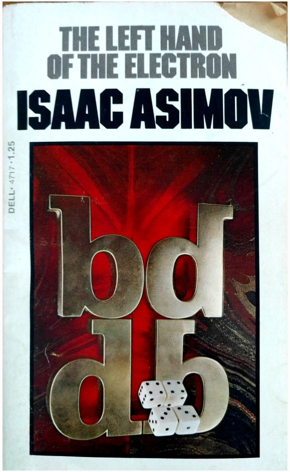
- P and T violation
- Chiral Superfluid ^{3}He-A
- Electrons in a Chiral Vacuum
- Electron Mobility in ^{3}He-A
- Scattering Theory
- $T \to 0$

- NSF Grant DMR-1508730

- H. Ikegami, Y. Tsutsumi, K. Kono , Science **341**, 59 (2013)

The Left Hand of the Electron, Issac Asimov, circa 1971

- An Essay on the Discovery of Parity Violation by the Weak Interaction



- ... And Reflections on Mirror Symmetry in Nature

Parity Violation in Beta Decay of ^{60}Co - Physical Review 105, 1413 (1957)

Experimental Test of Parity Conservation in Beta Decay*

C. S. Wu, *Columbia University, New York, New York*

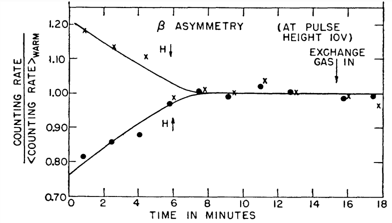
AND

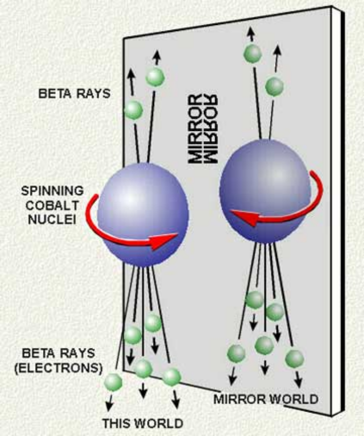
E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson,
National Bureau of Standards, Washington, D. C.

(Received January 15, 1957)

- T. D. Lee and C. N. Yang, Phys Rev 104, 204 (1956)

$$^{60}Co \rightarrow {}^{60}Ni + e^{-} + \bar{\nu}$$



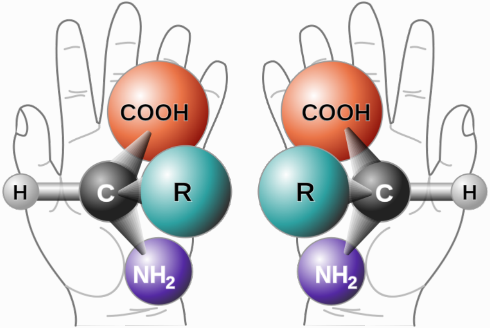


- Current of Beta electrons is (anti) correlated with the Spin of the ^{60}Co nucleus.

$$\langle \vec{S} \cdot \vec{p} \rangle \neq 0 \rightsquigarrow \text{Parity violation}$$

Chirality in Nature

Molecular Chiral Enantiomers



Handedness: Broken Mirror Symmetry

Chirality in Nature

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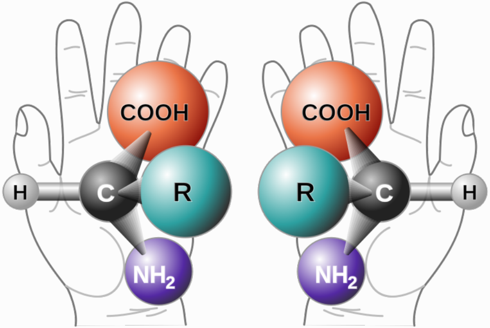
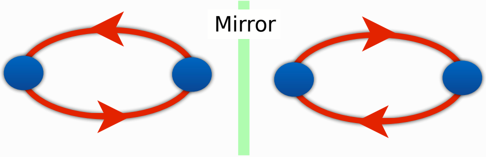


Handedness: Broken Mirror Symmetry

Chiral Diatomic Molecules

$$\Psi(\mathbf{r}) = f(r)\,(x + iy)$$



Broken Mirror Symmetries

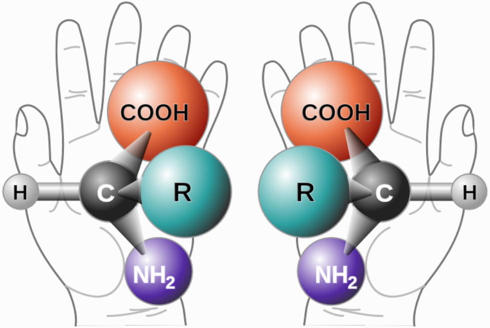
$$\Pi_{zx}\,\Psi(\mathbf{r}) = f(r)\,(x - iy)$$

Broken Time-Reversal Symmetry

$$\mathrm{T}\,\Psi(\mathbf{r}) = f(r)\,(x - iy)$$

Chirality in Nature

Molecular Chiral Enantiomers



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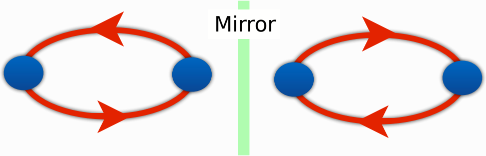

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Broken Time-Reversal Symmetry

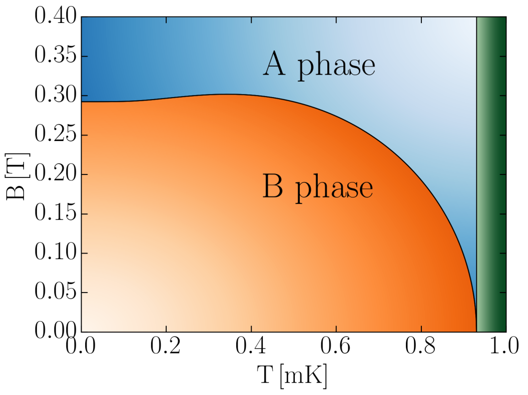
$$\mathrm{T}\,\Psi(\mathbf{r}) = f(r)\,(x - iy)$$

Realized in Superfluid ^{3}He-A

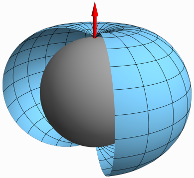
Superfluid Phases of ^{3}He in a Magnetic Field for $P < P_{\text{PCP}}$

Zeeman Energy for Spin-Triplet Pairing:

$$U_z = g_z \int d^3r\, B_\alpha \left[AA^\dagger\right]_{\alpha\beta} B_\beta$$



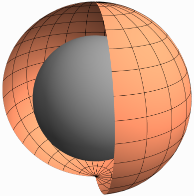
Chiral ABM State $\vec{l} = \hat{\mathbf{z}}$



$L_z = 1,\ S_z = 0$

$\mathbf{d}_z = \Delta\,(\hat{p}_x + i\hat{p}_y)$

"Isotropic" BW State



$J = 0,\ J_z = 0$

$\mathbf{d}_\alpha = \hat{p}_\alpha,\ \alpha = x, y, z$

Spin-Triplet, P-wave Order Parameter

$$\begin{pmatrix} \Psi_{\uparrow\uparrow} & \Psi_{\uparrow\downarrow} \\ \Psi_{\uparrow\downarrow} & \Psi_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_x + i\mathbf{d}_y & \mathbf{d}_z \\ \mathbf{d}_z & \mathbf{d}_x + i\mathbf{d}_y \end{pmatrix}$$

Signatures of Broken T and P Symmetry in ^{3}He-A

What is the Evidence for Chirality of Superfluid ^{3}He-A?

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Broken T and P $\rightsquigarrow$ Anomalous Hall Effect for Electrons in ^{3}He-A

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Broken T and P $\rightsquigarrow$ Anomalous Hall Effect for Electrons in ^{3}He-A

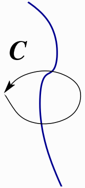
Spontaneous Symmetry Breaking $\rightsquigarrow$ Emergent Topology of ^{3}He-A

Chirality + Topology $\rightsquigarrow$ Weyl-Majorana Edge States & Chiral Edge Currents

Real-Space vs. Momentum-Space Topology

Topology in Real Space

$$\Psi(\mathbf{r}) = |\Psi(r)|\, e^{i\vartheta(\mathbf{r})}$$



Phase Winding

$$N_C = \frac{1}{2\pi} \oint_C d\vec{l}\cdot \frac{1}{|\Psi|} \mathrm{Im}[\boldsymbol{\nabla}\Psi] \in \{0, \pm 1, \pm 2, \ldots\}$$

- Massless Fermions confined in the Vortex Core

Real-Space vs. Momentum-Space Topology

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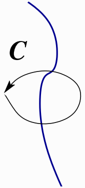
$\Psi(\mathbf{r}) = |\Psi(r)|\, e^{i\vartheta(\mathbf{r})}$



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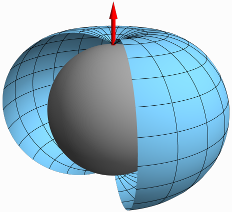
Chiral Symmetry $\rightsquigarrow$ Topology in Momentum Space

$\Psi(\mathbf{p}) = \Delta(p_x \pm i p_y) \sim e^{\pm i\varphi_{\mathbf{p}}}$



Topological Quantum Number: $L_z = \pm 1$

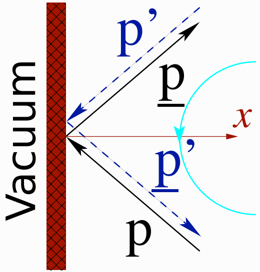
$$N_{2\mathrm{D}} = \frac{1}{2\pi}\oint d\mathbf{p}\cdot \frac{1}{|\Psi(\mathbf{p})|}\mathrm{Im}[\boldsymbol{\nabla}_{\mathbf{p}}\Psi(\mathbf{p})] = L_z$$

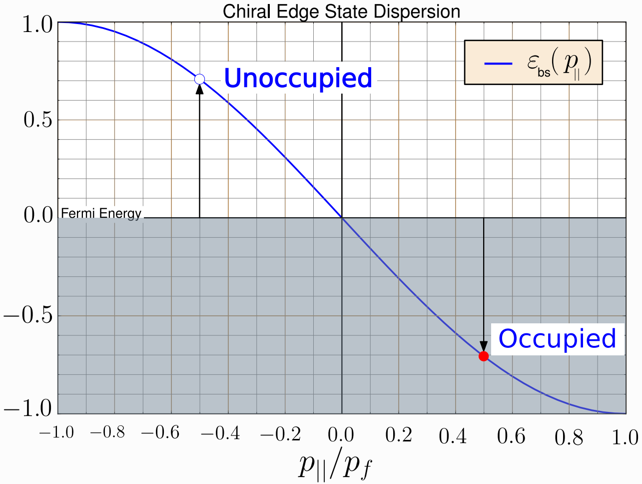
- Massless Chiral Fermions
  - Nodal Fermions in 3D
  - Edge Fermions in 2D

Massless Chiral Fermions in the 2D ^{3}He-A Films

Edge Fermions: $G^{R}_{\text{edge}}(\mathbf{p},\varepsilon;x) = \dfrac{\pi\Delta|\mathbf{p}_x|}{\varepsilon + i\gamma - \varepsilon_{\text{bs}}(\mathbf{p}_{||})}\, e^{-x/\xi_\Delta}$ $\quad \xi_\Delta = \hbar v_f/2\Delta \approx 10^2\,\mathring{A} \gg \hbar/p_f$

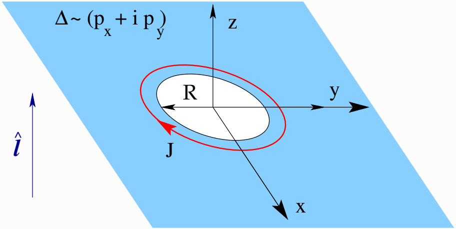
- $\varepsilon_{\text{bs}} = -c\,p_{||}$ with $c = \Delta/p_f \ll v_f$
- Broken P & T $\rightsquigarrow$ Edge Current





- J. A. Sauls, Phys. Rev. B 84, 214509 (2011)

Chiral Edge Current Circulating a Hole or Defect in a Chiral Superfluid



- $R \gg \xi_0 \approx 100\,\text{nm}$
- Sheet Current:

$$J \equiv \int dx\, J_{\varphi}(x)$$

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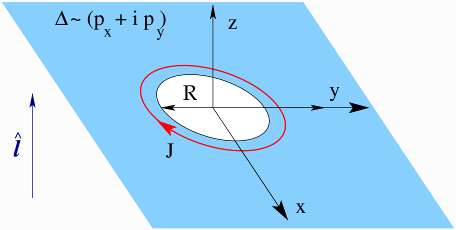

- $R \gg \xi_0 \approx 100\,\text{nm}$
- Sheet Current :
  $J \equiv \int dx\, J_{\varphi}(x)$

- Quantized Sheet Current: $\frac{1}{4}\,n\,\hbar$ ($n = N/V = {}^3$He density)
- Edge Current *Counter*-Circulates: $J = -\frac{1}{4}\,n\,\hbar$ w.r.t. Chirality: $\hat{\mathbf{l}} = +\mathbf{z}$

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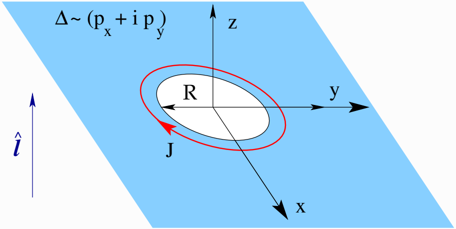

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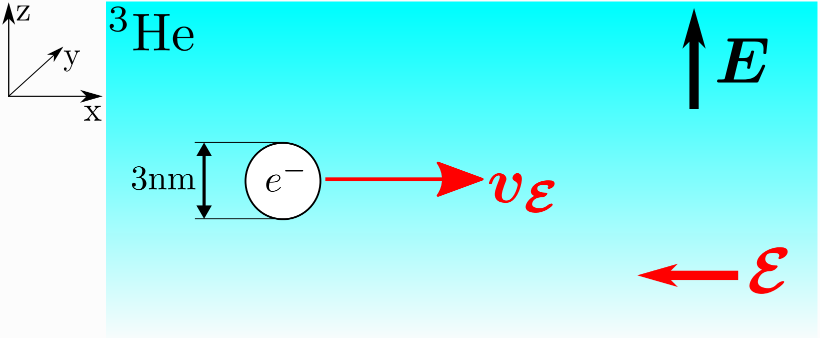
- Edge Current *Counter*-Circulates: $J = -\frac{1}{4}\, n\, \hbar$ w.r.t. Chirality: $\hat{\mathbf{l}} = +\mathbf{z}$
- Angular Momentum: $L_z = 2\pi\, h\, R^2 \times (-\frac{1}{4}\, n\, \hbar) = -(N_{\text{hole}}/2)\,\hbar$

N_{hole} = Number of ^{3}He atoms excluded from the Hole

∴ An object in ^{3}He-A *inherits* angular momentum from the Condensate of Chiral Pairs!

- J. A. Sauls, Phys. Rev. B 84, 214509 (2011)

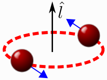
Electron bubbles in the Normal Fermi liquid phase of ^{3}He



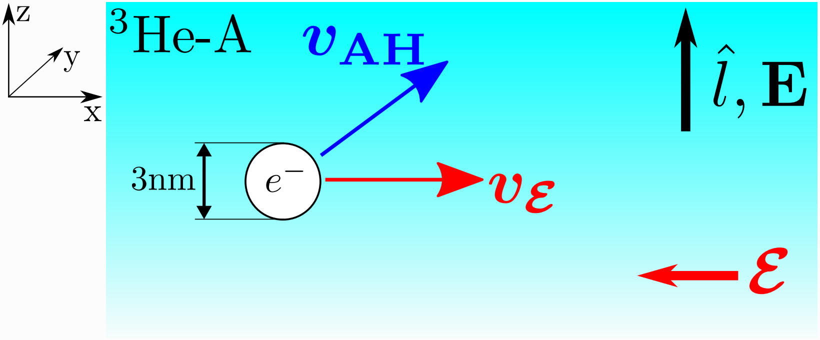
- Bubble with $R \simeq 1.5$ nm, $0.1\,\text{nm} \simeq \lambda_f \ll R \ll \xi_0 \simeq 80\,\text{nm}$
- Effective mass $M \simeq 100 m_3$ (m_3 – atomic mass of ^{3}He)
- QPs mean free path $l \gg R$
- Mobility of ^{3}He is *independent of* T for $T_c < T < 50$ mK

B. Josephson and J. Leckner, PRL 23, 111 (1969)

Electron bubbles in chiral superfluid ^{3}He-A

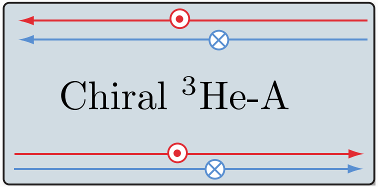


$\Delta(\hat{k}) = \Delta(\hat{k}_x + i\hat{k}_y) = \Delta\, e^{i\phi_{\mathbf{k}}}$

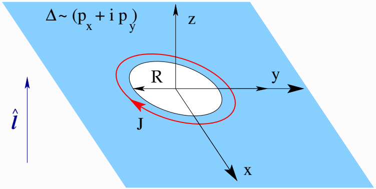


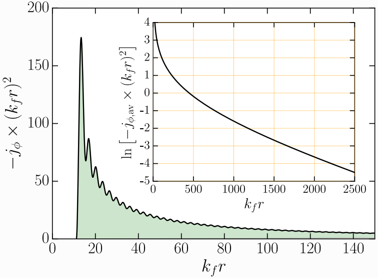
- Current: $\mathbf{v} = \overbrace{\mu_\perp \boldsymbol{\mathcal{E}}}^{\mathbf{v}_{\mathcal{E}}} + \overbrace{\mu_{\mathrm{AH}} \boldsymbol{\mathcal{E}} \times \hat{\mathbf{l}}}^{\mathbf{v}_{\mathrm{AH}}}$ R. Salmelin, M. Salomaa & V. Mineev, PRL **63**, 868 (1989)
- Hall ratio: $\tan\alpha = v_{\mathrm{AH}}/v_{\mathcal{E}} = |\mu_{\mathrm{AH}}/\mu_\perp|$

Current bound to an electron bubble ($k_f R = 11.17$)



$\Longrightarrow$

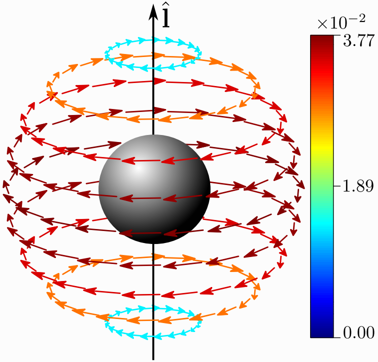




$$\mathbf{j}(\mathbf{r})/v_f N_f k_B T_c = j_\phi(\mathbf{r})\hat{\mathbf{e}}_\phi$$

- O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)

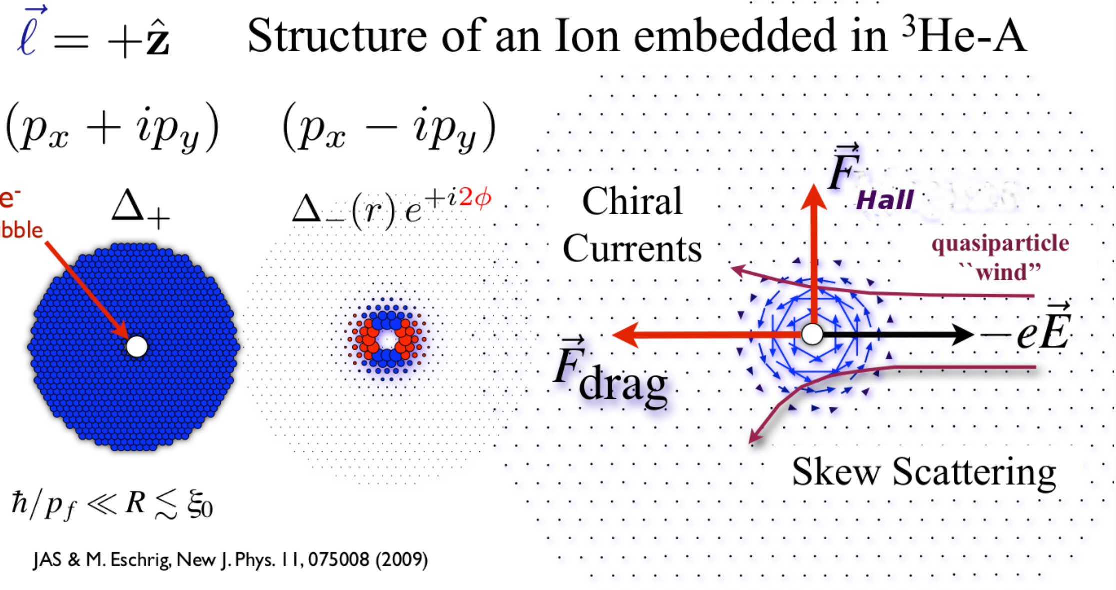
$\Longrightarrow$



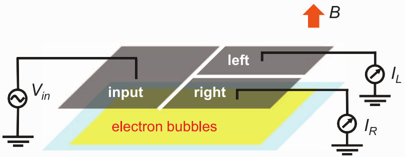
$$\mathbf{L}(T \to 0) \approx -\hbar N_{\text{bubble}}/2\,\hat{\mathbf{l}} \approx -100\,\hbar\,\hat{\mathbf{l}}$$

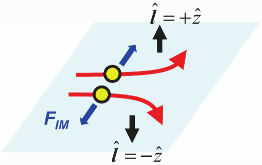
- JAS PRB **84**, 214509 (2011)

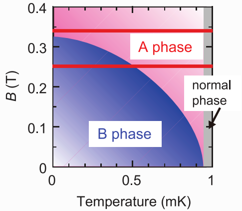
Skew Scattering of Quasiparticles by Bubble Edge Currents



Mobility of Electron Bubbles in ^{3}He-A

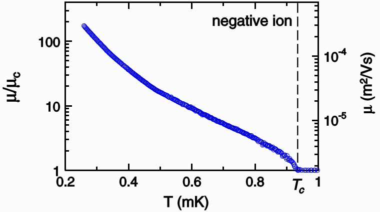


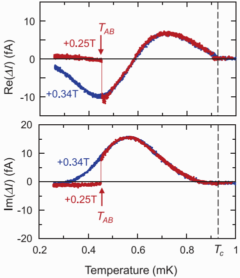


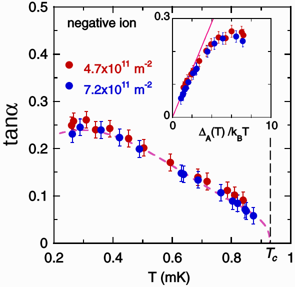


Electric current: $\mathbf{v} = \overbrace{\mu_\perp \boldsymbol{\mathcal{E}}}^{\mathbf{v}_{\mathcal{E}}} + \overbrace{\mu_{\mathrm{AH}} \boldsymbol{\mathcal{E}} \times \hat{\mathbf{l}}}^{\mathbf{v}_{\mathrm{AH}}}$

▶ Hall ratio: $\tan\alpha = v_{\mathrm{AH}}/v_{\mathcal{E}} = |\mu_{\mathrm{AH}}/\mu_\perp|$







▶ H. Ikegami et al., Science **341**, 59 (2013); JPSJ **82**, 124607 (2013); JPSJ **84**, 044602 (2015)

Forces on the Electron bubble in ^{3}He-A:

- $M\dfrac{d\mathbf{v}}{dt} = e\boldsymbol{\mathcal{E}} + \mathbf{F}_{\text{QP}}$, $\mathbf{F}_{QP}$ – force from quasiparticle collisions

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- $M\frac{d\mathbf{v}}{dt} = e\boldsymbol{\mathcal{E}} + \mathbf{F}_{\mathrm{QP}}$, $\mathbf{F}_{QP}$ – force from quasiparticle collisions
- $\mathbf{F}_{QP} = -\overleftrightarrow{\eta} \cdot \mathbf{v}$, $\overleftrightarrow{\eta}$ – generalized Stokes tensor
- $\overleftrightarrow{\eta} = \begin{pmatrix} \eta_{\perp} & \eta_{\mathsf{AH}} & 0 \\ -\eta_{\mathsf{AH}} & \eta_{\perp} & 0 \\ 0 & 0 & \eta_{\parallel} \end{pmatrix}$ for chiral symmetry with $\hat{\mathbf{l}} \parallel \mathbf{e}_z$

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- $M\dfrac{d\mathbf{v}}{dt} = e\boldsymbol{\mathcal{E}} - \eta_\perp\mathbf{v} + \dfrac{e}{c}\mathbf{v}\times\mathbf{B}_{\rm eff}$, for $\boldsymbol{\mathcal{E}} \perp \hat{\mathbf{l}}$
- $\mathbf{B}_{\rm eff} = -\dfrac{c}{e}\eta_{\sf AH}\hat{\mathbf{l}}$ $\quad B_{\rm eff} \simeq 10^3 - 10^4\ \rm T$!!!

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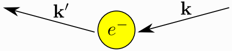
- $\mathbf{B}_{\mathrm{eff}} = -\frac{c}{e}\eta_{\mathsf{AH}}\hat{\mathbf{l}} \qquad B_{\mathrm{eff}} \simeq 10^3 - 10^4\ \mathrm{T}$!!!
- Mobility: $\frac{d\mathbf{v}}{dt} = 0 \quad \rightsquigarrow \quad \mathbf{v} = \overset{\leftrightarrow}{\mu}\boldsymbol{\mathcal{E}}$, where $\overset{\leftrightarrow}{\mu} = e\,\overset{\leftrightarrow}{\eta}^{-1}$

- O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)

T-matrix description of Quasiparticle-Ion scattering



- Lippmann-Schwinger equation for the T-matrix ($\varepsilon = E + i\eta\,;\ \eta \to 0^+$):

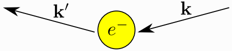
$$\hat{T}_S^R(\mathbf{k}', \mathbf{k}, E) = \hat{T}_N^R(\mathbf{k}', \mathbf{k}) + \int \frac{d^3k''}{(2\pi)^3} \hat{T}_N^R(\mathbf{k}', \mathbf{k}'') \Big[\hat{G}_S^R(\mathbf{k}'', E) - \hat{G}_N^R(\mathbf{k}'', E) \Big] \hat{T}_S^R(\mathbf{k}'', \mathbf{k}, E)$$

$$\hat{G}_S^R(\mathbf{k}, E) = \frac{1}{\varepsilon^2 - E_{\mathbf{k}}^2} \begin{pmatrix} \varepsilon + \xi_k & -\Delta(\hat{\mathbf{k}}) \\ -\Delta^\dagger(\hat{\mathbf{k}}) & \varepsilon - \xi_k \end{pmatrix}, \quad E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\Delta(\hat{\mathbf{k}})|^2}, \ \ \xi_k = \frac{\hbar^2 k^2}{2m^*} - \mu$$

- Normal-state T-matrix:

$$\hat{T}_N^R(\hat{\mathbf{k}}', \hat{\mathbf{k}}) = \begin{pmatrix} t_N^R(\hat{\mathbf{k}}', \hat{\mathbf{k}}) & 0 \\ 0 & -[t_N^R(-\hat{\mathbf{k}}', -\hat{\mathbf{k}})]^\dagger \end{pmatrix} \quad \text{in p-h (Nambu) space}$$

T-matrix description of Quasiparticle-Ion scattering



- Lippmann-Schwinger equation for the T-matrix ($\varepsilon = E + i\eta\,;\ \eta \to 0^+$):

$$\hat{T}_S^R(\mathbf{k}',\mathbf{k},E) = \hat{T}_N^R(\mathbf{k}',\mathbf{k}) + \int \frac{d^3k''}{(2\pi)^3} \hat{T}_N^R(\mathbf{k}',\mathbf{k}'') \left[\hat{G}_S^R(\mathbf{k}'',E) - \hat{G}_N^R(\mathbf{k}'',E)\right] \hat{T}_S^R(\mathbf{k}'',\mathbf{k},E)$$

$$\hat{G}_S^R(\mathbf{k},E) = \frac{1}{\varepsilon^2 - E_{\mathbf{k}}^2} \begin{pmatrix} \varepsilon + \xi_k & -\Delta(\hat{\mathbf{k}}) \\ -\Delta^\dagger(\hat{\mathbf{k}}) & \varepsilon - \xi_k \end{pmatrix}, \quad E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\Delta(\hat{\mathbf{k}})|^2}, \ \ \xi_k = \frac{\hbar^2 k^2}{2m^*} - \mu$$

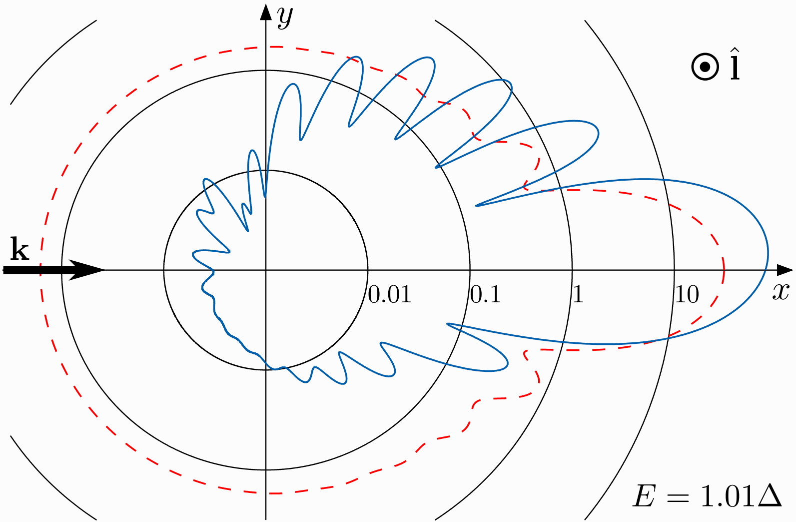
- Normal-state T-matrix:

$$\hat{T}_N^R(\hat{\mathbf{k}}',\hat{\mathbf{k}}) = \begin{pmatrix} t_N^R(\hat{\mathbf{k}}',\hat{\mathbf{k}}) & 0 \\ 0 & -[t_N^R(-\hat{\mathbf{k}}',-\hat{\mathbf{k}})]^\dagger \end{pmatrix}$$ in p-h (Nambu) space, where

$$t_N^R(\hat{\mathbf{k}}',\hat{\mathbf{k}}) = -\frac{1}{\pi N_f} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}), \quad P_l(x)$$ – Legendre function

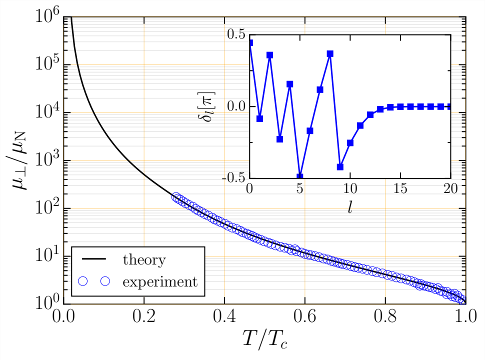
- Hard-sphere potential $\rightsquigarrow \tan\delta_l = j_l(k_f R)/n_l(k_f R)$ – spherical Bessel functions
  - $k_f R$ – determined by the Normal-State Mobility $\rightsquigarrow$ $k_f R = 11.17$ ($R = 1.42\,\mathrm{nm}$)

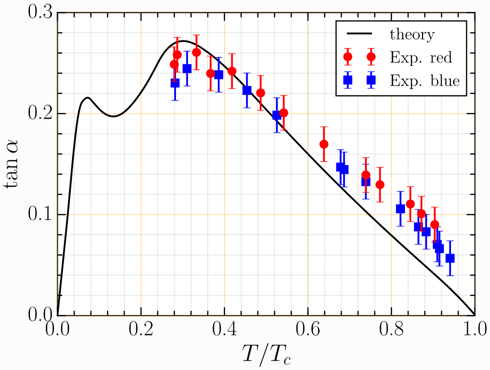
Differential cross section for Bogoliubov QP-Ion Scattering $k_f R = 11.17$



▶ O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)

Comparison between Theory and Experiment for the Drag and Transverse Forces





- $\mu_\perp = e\,\dfrac{\eta_\perp}{\eta_\perp^2 + \eta_{AH}^2}$
- $\mu_{AH} = -e\,\dfrac{\eta_{AH}}{\eta_\perp^2 + \eta_{AH}^2}$

- $\tan\alpha = \left|\dfrac{\mu_{AH}}{\mu_\perp}\right| = \dfrac{\eta_{AH}}{\eta_\perp}$
- Hard-Sphere Model: $k_f R = 11.17$

- O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)
- O. Shevtsov and JAS, JLTP 187, 340353 (2017)

Two Fluid Motion for a moving electron bubble as $T \to 0$

- An ion moving through a fluid experiences a force originating from the scattering of excitations off the ion.
- ^{3}He-A at $T \neq 0$ ^{3}He-A: a condensate of chiral Cooper pairs & a fluid of "normal" chiral Fermions.

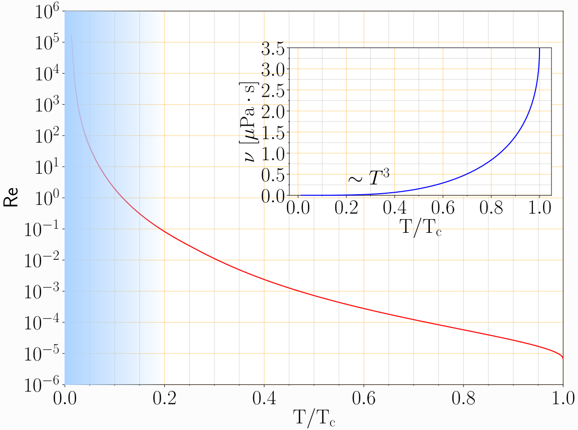
$$M \frac{d\mathbf{V}}{dt} = e\mathbf{E} + e\mathbf{V} \times \mathbf{B}_W - \eta \mathbf{V}$$

  - Dynamical Effective Mass of the Ion: M $\leftarrow$ Backflow & Virtual Excitations
  - Stokes Drag Force on the Ion: $\mathbf{F}_{\text{drag}} = -\eta\mathbf{V}$ $\leftarrow$ Dynamic Viscosity
  - Chiral Effective Magnetic Field: $\mathbf{B}_W = -\frac{c}{e}\eta_{xy}\hat{\mathbf{l}}$ $\leftarrow$ Anomalous Hall Response

- Stokes' drag for a sphere of radius R: $\eta = 6\pi\,\nu\,R \rightsquigarrow$ Reynold's Number: $\text{Re} \equiv \frac{2\rho V R}{\nu}$
- Normal ^{3}He: $\rho = 0.0819\,\text{g/cm}^3$ $\quad \mu_{\text{N}} = \frac{e}{n_3 p_f \sigma_{\text{N}}^{\text{tr}}} \simeq 1.7 \times 10^{-6}\,\text{m}^2/\text{V-s}$ $\quad \rightsquigarrow R = 1.42\,\text{nm}$ $\quad k_f R = 11.17$
- Derived Parameters: $\nu_{\text{N}} = \frac{\eta_{\text{N}}}{6\pi R} = 3.5 \times 10^{-6}\,\text{Pa-s}$ $\quad \text{Re}_{\text{N}} = 6.7 \times 10^{-6}$ $\quad \text{B}_{\text{N}} \equiv \frac{c}{e}\eta_{\text{N}} = 5.9 \times 10^5\,\text{T}$
- Reynold's Number for flow past an electron bubble in ^{3}He-A: $\text{Re} = \text{Re}_{\text{N}} \left(\frac{\eta_{\text{N}}}{\eta}\right)^{3/2} \xrightarrow[T \to 0]{} \sim \left(\frac{T_c}{T}\right)^{9/2}$!

Breakdown of Laminar Flow for $T \to 0$

$$\mathrm{Re} = \mathrm{Re}_\mathrm{N} \left(\frac{\eta_\mathrm{N}}{\eta}\right)^{3/2} \xrightarrow[T\to 0]{} \sim \left(\frac{T_c}{T}\right)^{9/2}$$



$$\mathrm{Re}_\mathrm{N} = 6.7 \times 10^{-6}$$

Vanishing of the Chiral Magnetic Field

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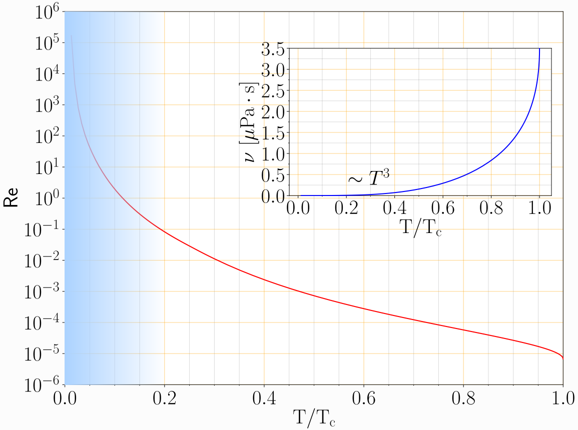
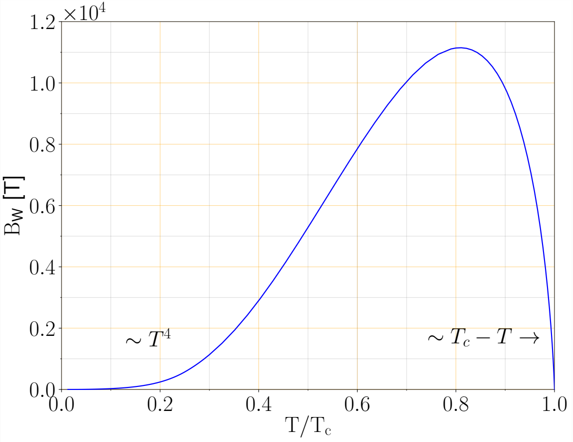


$$\mathrm{Re}_{\mathrm{N}} = 6.7 \times 10^{-6}$$

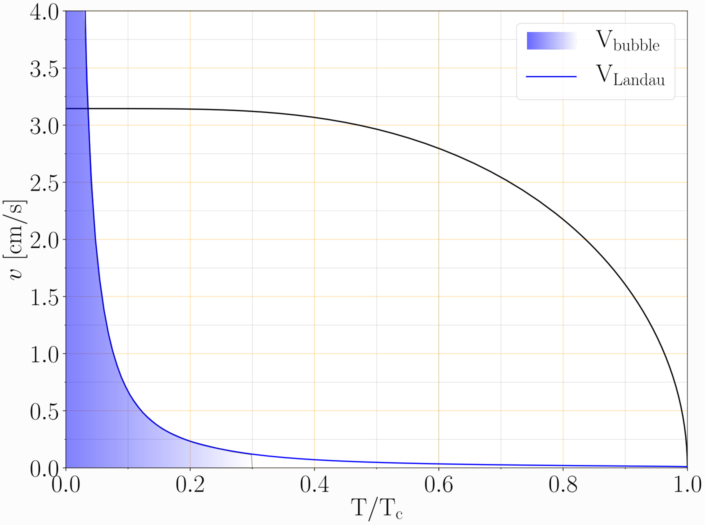
Vanishing of the Chiral Magnetic Field

$$B_{\mathrm{W}} = 5.9 \times 10^5\ \mathrm{T} \left(\frac{\eta_{xy}}{\eta_{\mathrm{N}}}\right)$$



$$\eta_{xy}/\eta_{\mathrm{N}}|_{T=0.8\,T_c} \approx \frac{\hbar}{p_f R}$$

Breakdown of Scattering Theory for $T \to 0$



Electron Bubble Velocity

- $V_{\mathsf{N}} = \mu_{\mathsf{N}} E_{\mathsf{N}} = 1.01 \times 10^{-4}$ m/s
- $V = \mu_{\mathsf{N}} E_{\mathsf{N}} \sqrt{\frac{\eta_{\mathsf{N}}}{\eta}}$

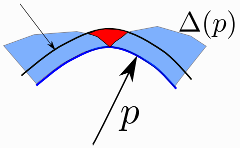
Maximum Landau critical velocity

- $V_c^{\text{max}} \approx 155 \times 10^{-4}$ m/s $\frac{\Delta_{\mathsf{A}}(T)}{k_b T_c}$

Nodal Superfluids:

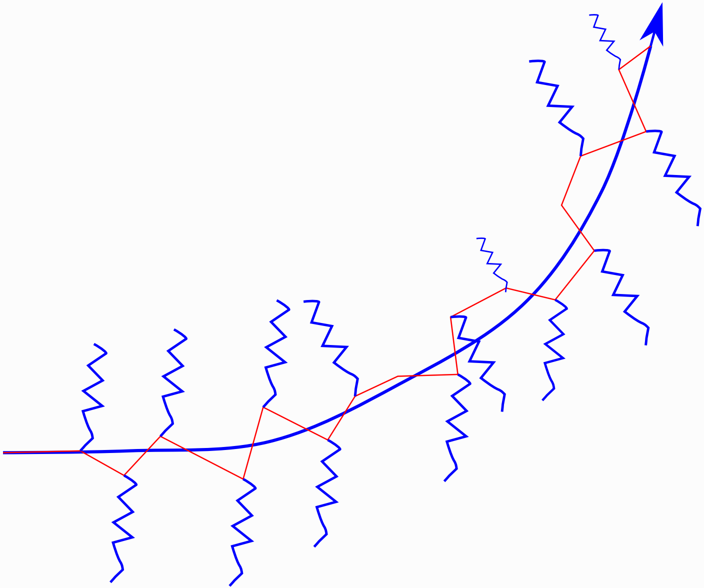
- $V_c = \Delta(p)/p_f \to 0$ for $p \to p_{\text{node}}$

$$\varepsilon_{\text{Doppler}} = p_f V$$



- Radiation Dominated Dampling for $T \lesssim 0.1 T_c$

Radiation Damping - Pair-Breaking



Feynman-Vernon Path-Integral Formulation of the Dynamics

- Ion + ^{3}He ("Bath" of quantum & thermal fluctuations) + ^{3}He-Ion Interation:
$$H = \frac{\mathbf{P}^2}{2M} - e\mathbf{E}\cdot\mathbf{R} + H_{\text{bath}} + H_{\text{int}}, \qquad H_{\text{int}}[\mathbf{R}] = \sum_{\alpha=\uparrow,\downarrow}\int d^3r\,\Psi^{\dagger}_{\alpha}(\mathbf{r})V(\mathbf{r}-\mathbf{R})\Psi_{\alpha}(\mathbf{r})$$

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- Reduced density matrix (RDM) for the Ion: $f(\mathbf{R},\mathbf{R}',t) = \langle\mathbf{R}|\hat{f}(t)|\mathbf{R}'\rangle = \mathrm{Tr}_{\text{bath}}\left\{\langle\mathbf{R}|\hat{\rho}(t)|\mathbf{R}'\rangle\right\}$,

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- Time Evolution of the Reduced Density Matrix:
$$f(\mathbf{R},\mathbf{R}',t) = \int d^3\text{R}_i \int d^3\text{R}'_i\; J(\mathbf{R},\mathbf{R}',t;\mathbf{R}_i,\mathbf{R}'_i,t_0)\, f_i(\mathbf{R}_i,\mathbf{R}'_i)$$

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- Path integral representation of the RDM propagator over forward $(\mathbf{R}_+)$ and backward $(\mathbf{R}_-)$ sub-paths:

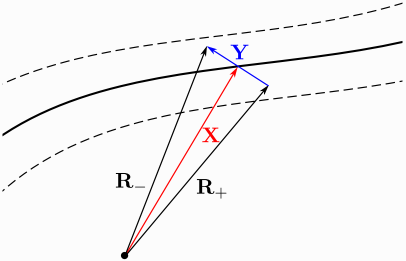
• $J(\mathbf{R},\mathbf{R}',t;\mathbf{R}_i,\mathbf{R}_i',t_0) = \int_{\mathbf{R}_i}^{\mathbf{R}}\mathfrak{D}\mathbf{R}_+\int_{\mathbf{R}_i'}^{\mathbf{R}'}\mathfrak{D}\mathbf{R}_-\,\exp\left\{\frac{i}{\hbar}\Big(S[\mathbf{R}_+]-S[\mathbf{R}_-]\Big)\right\}F[\mathbf{R}_+,\mathbf{R}_-]$,

• $S[\mathbf{R}] = \int_{t_0}^{t} d\tau\left[\frac{M\dot{\mathbf{R}}^2}{2} + e\mathbf{E}\cdot\mathbf{R}\right]$ – Action of a free Ion

• $F[\mathbf{R}_+,\mathbf{R}_-] = \mathrm{Tr}\left\{\hat{\rho}_T\,\hat{\mathcal{U}}^\dagger_{\rm bath}[\mathbf{R}_-;t,t_0]\,\hat{\mathcal{U}}_{\rm bath}[\mathbf{R}_+;t,t_0]\right\}$ – Feynman-Vernon influence functional

Stochastic Dynamics of an Heavy Ion in a Quantum Bath

- Forward and backward paths in RDM propagator:
  $\mathbf{X} = \dfrac{\mathbf{R}_+ + \mathbf{R}_-}{2} \rightsquigarrow$ "classical" trajectory,
  $\mathbf{Y} = \mathbf{R}_+ - \mathbf{R}_- \rightsquigarrow$ fluctuations around "classical" trajectory



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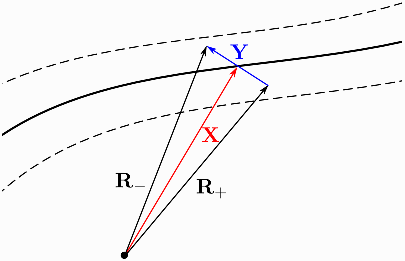

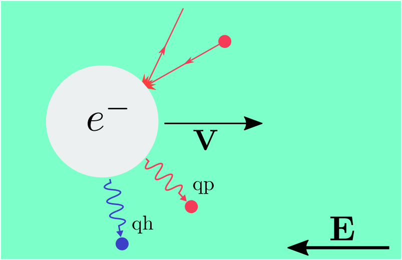

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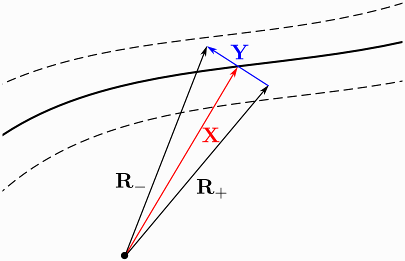
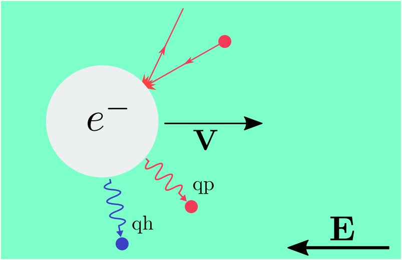
  $F[\mathbf{X}, \mathbf{Y}] \approx e^{i\Phi[\mathbf{X},\mathbf{Y}]}, \quad i\Phi[\mathbf{X}, \mathbf{Y}] = \underbrace{i\Phi_1[\mathbf{X}, \mathbf{Y}]}_{\sim O(\mathbf{Y})} - \underbrace{\Phi_2[\mathbf{X}, \mathbf{Y}]}_{\sim O(\mathbf{Y}^2)}$





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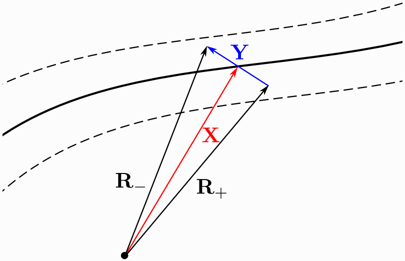
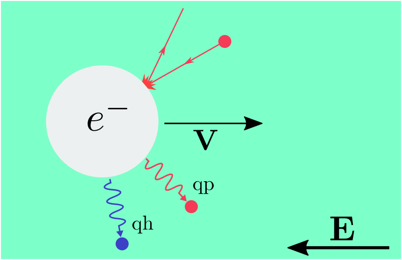
- Hubbard-Stratonovich transformation
  $$e^{-\Phi_2[\mathbf{X},\mathbf{Y}]} = \int \mathfrak{D}\boldsymbol{\xi}\, \mathfrak{F}[\boldsymbol{\xi}(t)] \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} d\tau \mathbf{Y}(\tau)\cdot\boldsymbol{\xi}(\tau)\right],$$

$\boldsymbol{\xi}(t) \rightsquigarrow$ Stochastic Force on the Ion





Stochastic Dynamics of an Heavy Ion in a Quantum Bath

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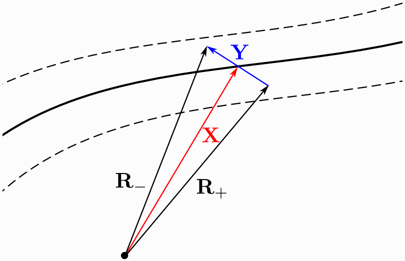
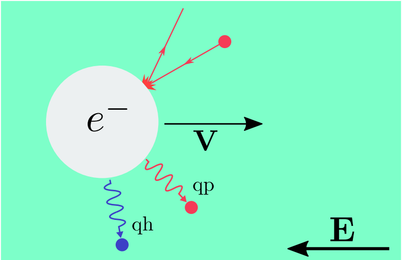
  $\boldsymbol{\xi}(t) \rightsquigarrow$ Stochastic Force on the Ion
- Stochastic field correlation function:
  $$S_{ab}^{[\mathbf{X}]}(t_1,t_2) = \left\langle K_a^{[\mathbf{X}]}(t_1)K_b^{[\mathbf{X}]}(t_2)\right\rangle - \left\langle K_a^{[\mathbf{X}]}(t_1)\right\rangle\left\langle K_b^{[\mathbf{X}]}(t_2)\right\rangle,$$
  $$\mathbf{K}^{[\mathbf{X}]}(t) = \hat{\mathcal{U}}^{\dagger}_{\text{bath}}[\mathbf{X};t,t_0]\left\{\nabla_{\mathbf{X}} H_{\text{int}}[\mathbf{X}]\right\}\hat{\mathcal{U}}_{\text{bath}}[\mathbf{X};t,t_0],$$
  $$\langle \xi_a(t_1)\xi_b(t_2)\rangle_\xi = 2\text{Re}S_{ab}^{[\mathbf{X}]}(t_1,t_2), \quad \langle\dots\rangle \equiv \text{tr}\{\hat\rho_T \dots\}$$





- After Hubbard-Stratonovich transformation the RDM propagator becomes:

$$J(\mathbf{R},\mathbf{R}',t;\mathbf{R}_i,\mathbf{R}_i',t_0) \propto \int \mathfrak{D}\boldsymbol{\xi}\ \mathfrak{F}[\boldsymbol{\xi}(t)] \int \mathfrak{D}\mathbf{X} \int \mathfrak{D}\mathbf{Y}$$
$$\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau \left[M\ddot{\mathbf{X}}\cdot\mathbf{Y} + |e|\mathbf{E}\cdot\mathbf{Y} + \mathbf{Y}\cdot\left\langle \mathbf{K}^{[\mathbf{X}]}(\tau)\right\rangle - \mathbf{Y}\cdot\boldsymbol{\xi}\right]\right\}$$

- Integrating out $\mathbf{Y}$ we arrive at the **Langevin equation**:

$$M\ddot{\mathbf{X}} + |e|\mathbf{E} + \left\langle \mathbf{K}^{[\mathbf{X}]}(t)\right\rangle = \boldsymbol{\xi}^{[\mathbf{X}]}(t)$$

  - self-consistent stochastic equation (mean force and stochastic field depend on $\mathbf{X}$)
  - describes semiclassical dynamics of the ion in presence of electric force and qp/qh scattering/emission

- Split ion's velocity into regular and fluctuating components:

$$\dot{\mathbf{X}}(t) = \mathbf{V} + \mathbf{v}(t) \quad \rightsquigarrow \quad M\dot{\mathbf{v}} + |e|\mathbf{E} + \underbrace{\left\langle \mathbf{K}^{[\mathbf{V}]}(t)\right\rangle}_{\text{static ion}} + \underbrace{\left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle}_{\sim O(|\mathbf{v}/v_f|,\boldsymbol{\xi})} + \underbrace{\left\langle \mathbf{K}_2^{[\mathbf{v}]}(t)\right\rangle}_{\sim O(|\mathbf{v}/v_f|^2,\boldsymbol{\xi}^2)} = \boldsymbol{\xi}^{[\mathbf{X}]}(t)$$

- Approximate solution scheme:

$$\left\langle\left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle\right\rangle_{\boldsymbol{\xi}} = 0: \quad M\dot{\mathbf{v}} + \left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle = \boldsymbol{\xi}^{[\mathbf{V}]}(t) \quad \xrightarrow{\boldsymbol{\xi}(t)} \quad |e|\mathbf{E} + \left\langle \mathbf{K}^{[\mathbf{V}]}(t)\right\rangle + \left\langle\left\langle \mathbf{K}_2^{[\mathbf{v}]}(t)\right\rangle\right\rangle_{\boldsymbol{\xi}} = \mathbf{0}$$

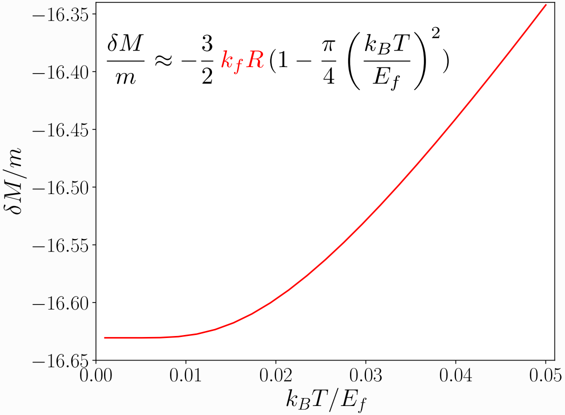
Solution to the Langevin equation allows to extract the effective mass:

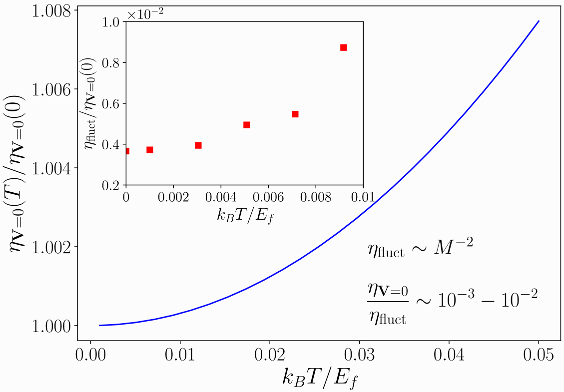
$$M\dot{v}_i + \int_{-\infty}^{t} d\tau\, \chi_{ij}(t-\tau)v_j(\tau) = \xi_i^{[\mathbf{V}]}(t), \quad \chi(\omega) \xrightarrow{\omega\to 0} -i\omega\delta M \quad \rightsquigarrow \quad M_{\text{eff}} = M + \delta M$$

Low velocity limit: i.e. small applied field $\mathbf{E}$: $\quad e\mathbf{E} - \eta_{\text{tot}}\mathbf{V} = 0,$

$$\eta_{\text{tot}}(T,M,R) = \eta_{\mathbf{V}=0}(T,R) + \eta_{\text{fluct}}(T,M,R), \quad \eta_{\mathbf{V}=0}(T\to 0) = n_3 p_f \sigma_N^{tr}(p_f)$$

Josephson-Leckner (1969)





Summary

- Electrons in ^{3}He-A are "dressed" by a spectrum of Weyl Fermions
- Electrons in ^{3}He-A are "Left handed" in a Right-handed Chiral Vacuum
  $\rightsquigarrow L_z \approx -(N_{bubble}/2)\hbar \approx -100\,\hbar$
- Experiment: RIKEN mobility experiments $\rightsquigarrow$ Observation an AHE in ^{3}He-A
- Scattering of Bogoliubov QPs by the dressed Ion
  $\rightsquigarrow$ Drag Force $(-\eta_{\perp}\mathbf{v})$ and Transverse Force $(\frac{e}{c}\mathbf{v} \times \mathbf{B}_{eff})$ on the Ion
- *Anomalous Hall Field:* $\mathbf{B}_{\text{eff}} \approx \frac{\Phi_0}{3\pi^2}\, k_f^2\,(k_f R)^2 \left(\frac{\eta_{\text{AH}}}{\eta_{\text{N}}}\right) \mathbf{l} \simeq 10^3 - 10^4\,\text{T}\,\mathbf{l}$
- Mechanism: Skew/Andreev Scattering of Bogoliubov QPs by the dressed Ion
- Origin: Broken Mirror & Time-Reversal Symmetry $\rightsquigarrow W(\mathbf{k},\mathbf{k}') \neq W(\mathbf{k}',\mathbf{k})$
- Theory: $\rightsquigarrow$ Quantitative account of RIKEN mobility experiments
- Ongoing: Radiation Dominated Motion of Electrons and Ions in a Chiral Vacuum
- New directions for Transport in ^{3}He-A & Chiral Superconductors

- After Hubbard-Stratonovich transformation the RDM propagator becomes:

$$J(\mathbf{R}, \mathbf{R}', t; \mathbf{R}_i, \mathbf{R}'_i, t_0) \propto \int \mathfrak{D}\boldsymbol{\xi}\; \mathfrak{F}[\boldsymbol{\xi}(t)] \int \mathfrak{D}\mathbf{X} \int \mathfrak{D}\mathbf{Y}$$

$$\exp\left\{ -\frac{i}{\hbar} \int_{t_0}^{t} d\tau \left[M\ddot{\mathbf{X}} \cdot \mathbf{Y} + |e|\mathbf{E} \cdot \mathbf{Y} + \mathbf{Y} \cdot \left\langle \mathbf{K}^{[\mathbf{X}]}(\tau) \right\rangle - \mathbf{Y} \cdot \boldsymbol{\xi} \right] \right\}$$

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$$\dot{\mathbf{X}}(t) = \mathbf{V} + \mathbf{v}(t) \quad \rightsquigarrow \quad M\dot{\mathbf{v}} + |e|\mathbf{E} + \underbrace{\left\langle \mathbf{K}^{[\mathbf{V}]}(t)\right\rangle}_{\text{static ion}} + \underbrace{\left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle}_{\sim O(|\mathbf{v}/v_f|,\boldsymbol{\xi})} + \underbrace{\left\langle \mathbf{K}_2^{[\mathbf{v}]}(t)\right\rangle}_{\sim O(|\mathbf{v}/v_f|^2,\boldsymbol{\xi}^2)} = \boldsymbol{\xi}^{[\mathbf{X}]}(t)$$

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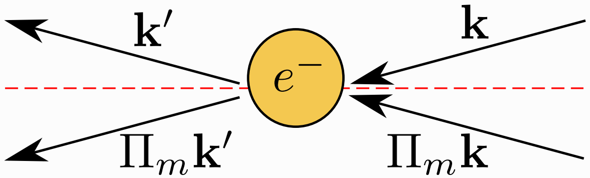
- Approximate solution scheme:

$$\left\langle\left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle\right\rangle_{\boldsymbol{\xi}} = 0: \quad M\dot{\mathbf{v}} + \left\langle \mathbf{K}_1^{[\mathbf{v}]}(t)\right\rangle = \boldsymbol{\xi}^{[\mathbf{V}]}(t) \quad \xrightarrow{\boldsymbol{\xi}(t)} \quad |e|\mathbf{E} + \left\langle \mathbf{K}^{[\mathbf{V}]}(t)\right\rangle + \left\langle\left\langle \mathbf{K}_2^{[\mathbf{v}]}(t)\right\rangle\right\rangle_{\boldsymbol{\xi}} = \mathbf{0}$$

Broken Time-Reversal ($\mathtt{T}$) & mirror (Π_{m}) symmetries in Chiral Superfluids



- Broken TRS: $\mathtt{T} \cdot (\hat{p}_x + i\hat{p}_y) = (\hat{p}_x - i\hat{p}_y)$
- Broken mirror symmetry: $\Pi_{\mathrm{m}} \cdot (\hat{p}_x + i\hat{p}_y) = (\hat{p}_x - i\hat{p}_y)$
- Chiral symmetry: $\mathtt{C} = \mathtt{T} \times \Pi_{\mathrm{m}} \quad \rightsquigarrow \quad \mathtt{C} \cdot (\hat{p}_x + i\hat{p}_y) = (\hat{p}_x + i\hat{p}_y)$
- Microscopic reversibility for chiral superfluids: $W(\hat{\mathbf{k}}', \hat{\mathbf{k}}; +\hat{\mathbf{l}}) = W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; -\hat{\mathbf{l}})$
- $\therefore$ For BTRS: the chiral axis $\hat{\mathbf{l}}$ is fixed $\rightsquigarrow$ $W(\hat{\mathbf{k}}', \hat{\mathbf{k}}; \hat{\mathbf{l}}) \neq W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \hat{\mathbf{l}})$

Confinement: Superfluid Phases of ^{3}He in Thin Films

Symmetry or Normal Liquid ^{3}*He*: $\mathtt{G} = \mathtt{SO(3)_S} \times \mathtt{SO(2)_L} \times \mathtt{U(1)_N} \times \mathtt{P} \times \mathtt{T}$

- Length Scale for Strong Confinement:
  $\xi_0 = \hbar v_f / 2\pi k_B T_c \approx 20 - 80\,\text{nm}$

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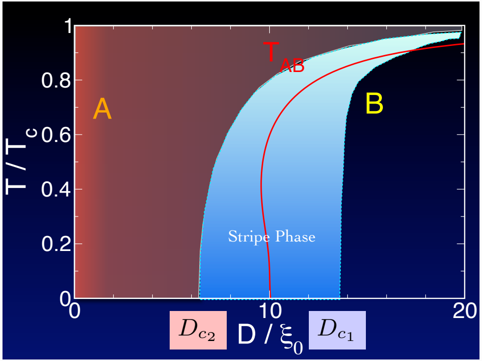
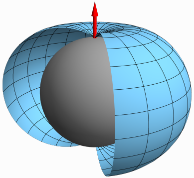
A. Vorontsov & JAS, PRL, 2007

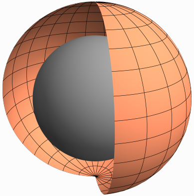


Chiral AM State $\vec{l} = \hat{\mathbf{z}}$



$L_z = 1,\ S_z = 0$

"Isotropic" BW State



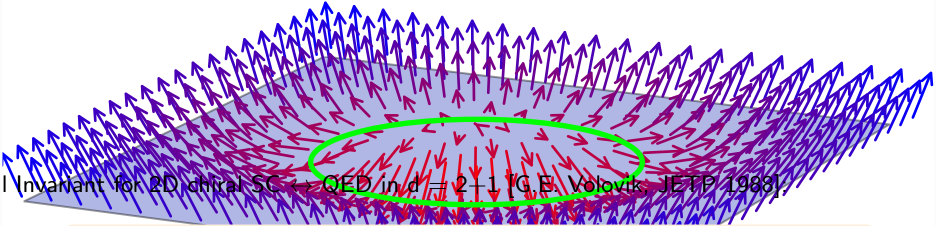
$J = 0,\ J_z = 0$

Hamiltonian for quasi-2D Chiral Superconductor (Sr_2RuO_4 & ^{3}He-A Film):

⇝ Momentum-Space Topology of Nambu-Bogoliubov H

$$\widehat{H} = \begin{pmatrix} (|\mathbf{p}|^2/2m^* - \mu) & c(p_x + ip_y) \\ c(p_x - ip_y) & -(|\mathbf{p}|^2/2m^* - \mu) \end{pmatrix} = \vec{\mathbf{m}}(\mathbf{p}) \cdot \widehat{\vec{\boldsymbol{\tau}}}$$

$$\vec{\mathbf{m}} = (\, cp_x \,, \mp cp_y \,, \xi(\mathbf{p})) \quad \text{with } |\vec{\mathbf{m}}(\mathbf{p})|^2 = (|\mathbf{p}|^2/2m - \mu)^2 + c^2|\mathbf{p}|^2 > 0 \,, \mu \neq 0$$



Topological Invariant for 2D chiral SC ↔ QED in d = 2+1 [G.E. Volovik, JETP 1988]:

$$N_{2D} = \pi \int \frac{d^2p}{(2\pi)^2}\, \hat{\mathbf{m}}(\mathbf{p}) \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial p_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial p_y} \right) = \begin{cases} \pm 1\,; & \mu > 0 \text{ and } \Delta \neq 0 \\ 0\,; & \mu < 0 \text{ or } \Delta = 0 \end{cases}$$

"Vacuum" $(\Delta = 0)$ with $N_{2D} = 0$ | ^{3}He-A $(\Delta \neq 0)$ with $N_{2D} = 1$

Zero Energy Fermions ↑ Confined on the Edge

Determination of the Electron Bubble Radius

(i) Energy required to create a bubble:

$$E(R,P) = E_0(U_0,R) + 4\pi R^2\gamma + \frac{4\pi}{3}R^3 P, \quad P \text{ – pressure}$$

(ii) For $U_0 \to \infty$: $E_0 = -U_0 + \pi^2\hbar^2/2m_e R^2$ – ground state energy

(iii) Surface Energy: hydrostatic surface tension $\rightsquigarrow \gamma = 0.15\,\mathrm{erg/cm^2}$

(iv) Minimizing E w.r.t. $R \rightsquigarrow P = \pi\hbar^2/4m_e R^5 - 2\gamma/R$

(v) For zero pressure, $P = 0$:

$$R = \left(\frac{\pi\hbar^2}{8m_e\gamma}\right)^{1/4} \approx 2.38\,\mathrm{nm} \quad \rightsquigarrow \quad k_f R = 18.67$$

Transport $\rightsquigarrow k_f R = 11.17$

- A. Ahonen et al., J. Low Temp. Phys., 30(1):205228, 1978

Calculation of LDOS and Current Density

$$\hat{\mathcal{G}}_S^R(\mathbf{r}',\mathbf{r},E) = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} e^{i\mathbf{k}'\mathbf{r}'} e^{-i\mathbf{k}\mathbf{r}} \hat{\mathcal{G}}_S^R(\mathbf{k}',\mathbf{k},E)$$

$$\hat{\mathcal{G}}_S^R(\mathbf{k}',\mathbf{k},E) = (2\pi)^3 \hat{G}_S^R(\mathbf{k},E)\delta_{\mathbf{k}',\mathbf{k}} + \hat{G}_S^R(\mathbf{k}',E)\hat{T}_S(\mathbf{k}',\mathbf{k},E)\hat{G}_S^R(\mathbf{k},E)$$

$$\hat{G}_S^R(\mathbf{k},E) = \frac{1}{\varepsilon^2 - E_{\mathbf{k}}^2}\begin{pmatrix} \varepsilon + \xi_k & -\Delta(\hat{\mathbf{k}}) \\ -\Delta^\dagger(\hat{\mathbf{k}}) & \varepsilon - \xi_k \end{pmatrix}, \quad \varepsilon = E + i\eta, \quad \eta \to 0^+$$

$$N(\mathbf{r},E) = -\frac{1}{2\pi}\mathrm{Im}\left\{\mathrm{Tr}\left[\hat{\mathcal{G}}_S^R(\mathbf{r},\mathbf{r},E)\right]\right\}$$

$$\mathbf{j}(\mathbf{r}) = \frac{\hbar}{4mi} k_B T \sum_{n=-\infty}^{\infty} \lim_{\mathbf{r}\to\mathbf{r}'} \mathrm{Tr}\left[(\boldsymbol{\nabla}_{\mathbf{r}'} - \boldsymbol{\nabla}_{\mathbf{r}})\hat{\mathcal{G}}^M(\mathbf{r}',\mathbf{r},\epsilon_n)\right]$$

$$\hat{\mathcal{G}}_S^R(\mathbf{r}',\mathbf{r},E) = \hat{\mathcal{G}}_S^M(\mathbf{r}',\mathbf{r},\epsilon_n)\Big|_{i\epsilon_n\to\varepsilon}, \text{ for } n \geq 0$$

$$\hat{\mathcal{G}}_S^M(\mathbf{k},\mathbf{k}',-\epsilon_n) = \left[\hat{\mathcal{G}}_S^M(\mathbf{k}',\mathbf{k},\epsilon_n)\right]^\dagger$$

Weyl Fermion Spectrum bound to the Electron Bubble

$$\mu_{\mathsf{N}} = \frac{e}{n_3 p_f \sigma_{\mathsf{N}}^{\mathsf{tr}}} \quad \Leftarrow \quad \mu_{\mathsf{N}}^{\mathsf{exp}} = 1.7 \times 10^{-6} \, \frac{m^2}{V\,s}$$

$$\tan \delta_l = j_l(k_f R)/n_l(k_f R) \quad \Rightarrow \quad \sigma_{\mathsf{N}}^{\mathsf{tr}} = \frac{4\pi}{k_f^2} \sum_{l=0}^{\infty} (l+1) \sin^2(\delta_{l+1} - \delta_l) \quad \rightsquigarrow \quad \boxed{k_f R = 11.17}$$

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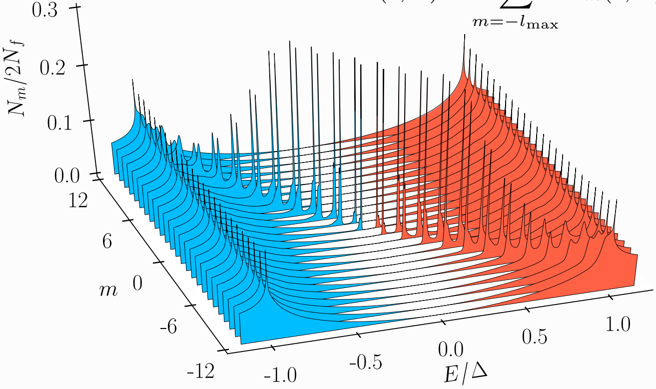
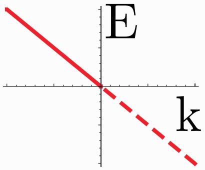
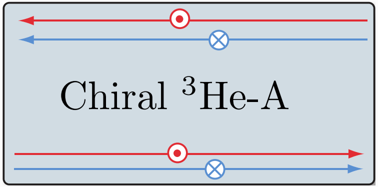
$$N(\mathbf{r}, E) = \sum_{m=-l_{\max}}^{l_{\max}} N_m(\mathbf{r}, E), \quad l_{\max} \simeq k_f R$$

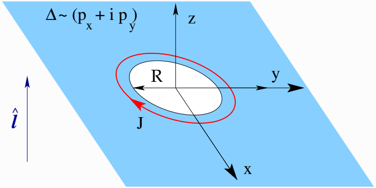


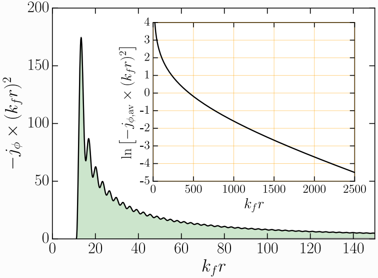


Current bound to an electron bubble ($k_f R = 11.17$)



$\Longrightarrow$

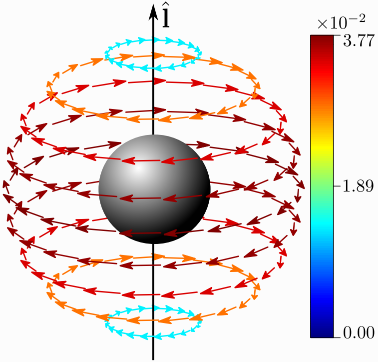




$$\mathbf{j}(\mathbf{r})/v_f N_f k_B T_c = j_\phi(\mathbf{r})\hat{\mathbf{e}}_\phi$$

- O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)

$\Longrightarrow$

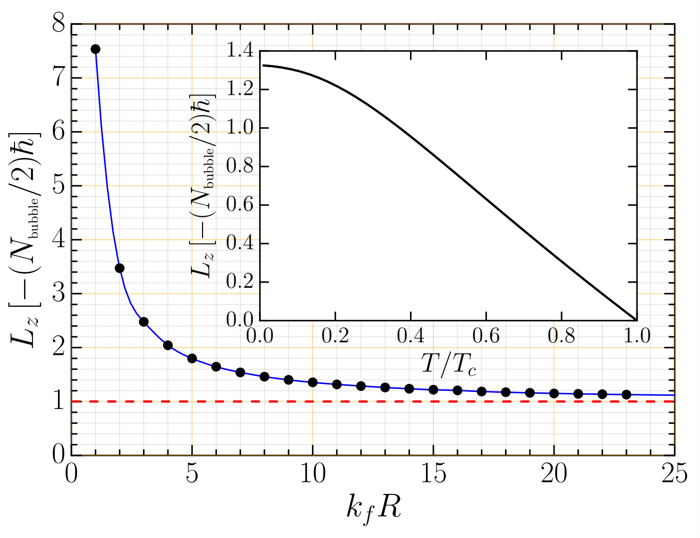


$$\mathbf{L}(T \to 0) \approx -\hbar N_{\text{bubble}}/2\, \hat{\mathbf{l}} \approx -100\, \hbar\, \hat{\mathbf{l}}$$

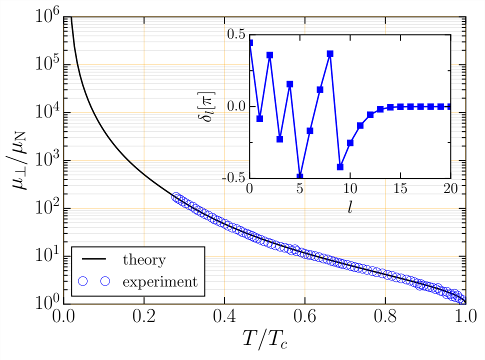
- JAS PRB **84**, 214509 (2011)

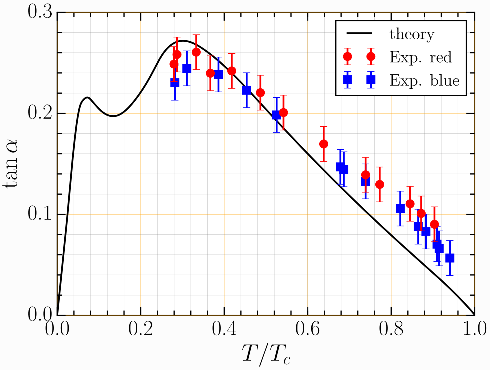
Angular momentum of an electron bubble in ^{3}He-A ($k_f R = 11.17$)

$$\mathbf{L}(T \to 0) \approx -\hbar N_{\text{bubble}} \hat{\mathbf{l}}/2 \,; \qquad N_{\text{bubble}} = n_3 \, \frac{4\pi}{3} R^3 \approx 200 \ {}^3\text{He atoms}$$



Comparison between Theory and Experiment for the Drag and Transverse Forces





- $\mu_\perp = e\,\dfrac{\eta_\perp}{\eta_\perp^2 + \eta_{\text{AH}}^2}$
- $\mu_{\text{AH}} = -e\,\dfrac{\eta_{\text{AH}}}{\eta_\perp^2 + \eta_{\text{AH}}^2}$

- $\tan\alpha = \left|\dfrac{\mu_{\text{AH}}}{\mu_\perp}\right| = \dfrac{\eta_{\text{AH}}}{\eta_\perp}$
- Hard-Sphere Model: $k_f R = 11.17$

- O. Shevtsov and JAS, Phys. Rev. B 96, 064511 (2016)
- O. Shevtsov and JAS, JLTP 187, 340353 (2017)

Theoretical Models for the QP-ion potential

- $U(r) = \begin{cases} U_0, & r < R, \\ -U_1, & R < r < R', \\ 0, & r > R'. \end{cases}$

- $\rightsquigarrow$ Hard-Sphere Potential: $U_1 = 0$, $R' = R$, $U_0 \to \infty$

- $U(x) = U_0\,[1 - \tanh[(x-b)/c]]$, $\quad x = k_f r$

- $U(x) = U_0/\cosh^2[\alpha x^n]$, $\quad x = k_f r$ (Pöschl-Teller-like potential)

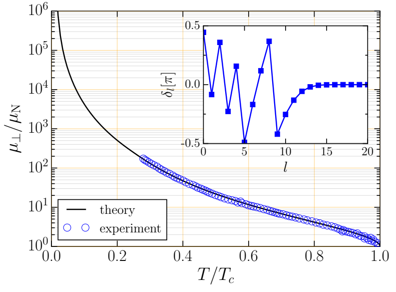
- Random phase shifts: $\{\delta_l | \, l = 1 \ldots l_{\max}\}$ are generated with δ_0 is an adjustable parameter

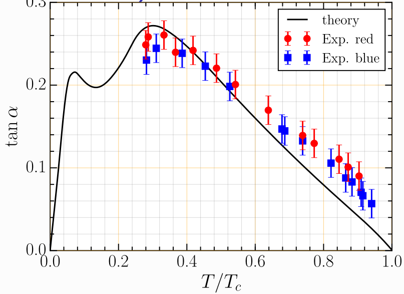
- Parameters for all models are chosen to fit the experimental value of the normal-state mobility, $\mu_N^{\exp} = 1.7 \times 10^{-6}\, m^2/V \cdot s$

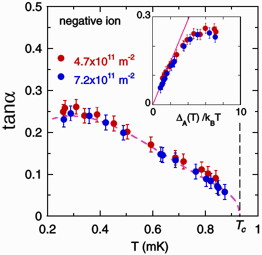
Theoretical Models for the QP-ion potential

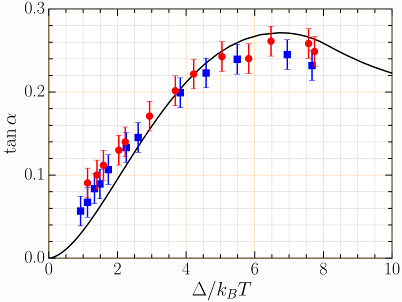
Label	Potential	Parameters
Model A	hard sphere	$k_f R = 11.17$
Model B	repulsive core & attractive well	$U_0 = 100E_f, U_1 = 10E_f, k_f R' = 11, R/R' = 0.36$
Model C	random phase shifts model 1	$l_{\max} = 11$
Model D	random phase shifts model 2	$l_{\max} = 11$
Model E	Pöschl-Teller-like	$U_0 = 1.01E_f, k_f R = 22.15, \alpha = 3 \times 10^{-5}, n = 4$
Model F	Pöschl-Teller-like	$U_0 = 2E_f, k_f R = 19.28, \alpha = 6 \times 10^{-5}, n = 4$
Model G	hyperbolic tangent	$U_0 = 1.01E_f, k_f R = 14.93, b = 12.47, c = 0.246$
Model H	hyperbolic tangent	$U_0 = 2E_f, k_f R = 14.18, b = 11.92, c = 0.226$
Model I	soft sphere 1	$U_0 = 1.01E_f, k_f R = 12.48$
Model J	soft sphere 2	$U_0 = 2E_f, k_f R = 11.95$

Hard-sphere model with $k_f R = 11.17$ (Model A)



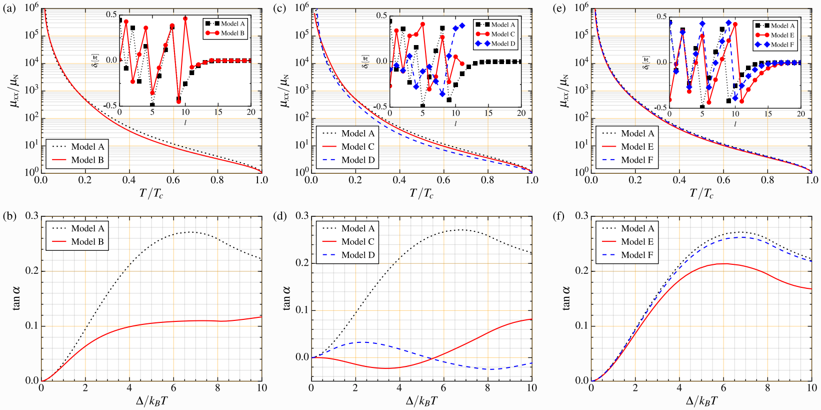






Comparison with Experiment for Models for the QP-ion potential

Label	Potential	Parameters
Model A	hard sphere	$k_f R = 11.17$
Model B	attractive well with a repulsive core	$U_0 = 100E_f, U_1 = 10E_f, k_f R' = 11, R/R' = 0.36$
Model C	random phase shifts model 1	$l_{\max} = 11$
Model D	random phase shifts model 2	$l_{\max} = 11$
Model E	Pöschl-Teller-like	$U_0 = 1.01E_f, k_f R = 22.15, \alpha = 3 \times 10^{-5}, n = 4$
Model F	Pöschl-Teller-like	$U_0 = 2E_f, k_f R = 19.28, \alpha = 6 \times 10^{-5}, n = 4$



Mobility of an electron bubble in the Normal Fermi Liquid

(i) $t_{\mathsf{N}}^{R}(\hat{\mathbf{k}}',\hat{\mathbf{k}};E)=\sum_{l=0}^{\infty}(2l+1)t_{l}^{R}(E)P_{l}(\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}})$

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Mobility of an electron bubble in the Normal Fermi Liquid

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(v) $$\mu_{\mathsf{N}} = \frac{e}{n_3 p_f \sigma_{\mathsf{N}}^{\mathsf{tr}}}, \quad p_f = \hbar k_f, \quad n_3 = \frac{k_f^3}{3\pi^2}$$

Calculation of LDOS and Current Density

$$\hat{\mathcal{G}}_S^R(\mathbf{r}', \mathbf{r}, E) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{i\mathbf{k}'\mathbf{r}'} e^{-i\mathbf{k}\mathbf{r}} \hat{\mathcal{G}}_S^R(\mathbf{k}', \mathbf{k}, E)$$

$$\hat{\mathcal{G}}_S^R(\mathbf{k}', \mathbf{k}, E) = (2\pi)^3 \hat{G}_S^R(\mathbf{k}, E)\delta_{\mathbf{k}',\mathbf{k}} + \hat{G}_S^R(\mathbf{k}', E)\hat{T}_S(\mathbf{k}', \mathbf{k}, E)\hat{G}_S^R(\mathbf{k}, E)$$

$$\hat{G}_S^R(\mathbf{k}, E) = \frac{1}{\varepsilon^2 - E_{\mathbf{k}}^2} \begin{pmatrix} \varepsilon + \xi_k & -\Delta(\hat{\mathbf{k}}) \\ -\Delta^\dagger(\hat{\mathbf{k}}) & \varepsilon - \xi_k \end{pmatrix}, \quad \varepsilon = E + i\eta, \quad \eta \to 0^+$$

$$N(\mathbf{r}, E) = -\frac{1}{2\pi} \mathrm{Im} \left\{ \mathrm{Tr} \left[\hat{\mathcal{G}}_S^R(\mathbf{r}, \mathbf{r}, E) \right] \right\}$$

$$\mathbf{j}(\mathbf{r}) = \frac{\hbar}{4mi} k_B T \sum_{n=-\infty}^{\infty} \lim_{\mathbf{r}\to\mathbf{r}'} \mathrm{Tr} \left[(\boldsymbol{\nabla}_{\mathbf{r}'} - \boldsymbol{\nabla}_{\mathbf{r}}) \hat{\mathcal{G}}^M(\mathbf{r}', \mathbf{r}, \epsilon_n) \right]$$

$$\hat{\mathcal{G}}_S^R(\mathbf{r}', \mathbf{r}, E) = \hat{\mathcal{G}}_S^M(\mathbf{r}', \mathbf{r}, \epsilon_n) \Big|_{i\epsilon_n \to \varepsilon}, \ \text{for } n \geq 0$$

$$\hat{\mathcal{G}}_S^M(\mathbf{k}, \mathbf{k}', -\epsilon_n) = \left[\hat{\mathcal{G}}_S^M(\mathbf{k}', \mathbf{k}, \epsilon_n) \right]^\dagger$$

Temperature scaling of the Stokes tensor components

- For $1 - \dfrac{T}{T_c} \to 0^+$:

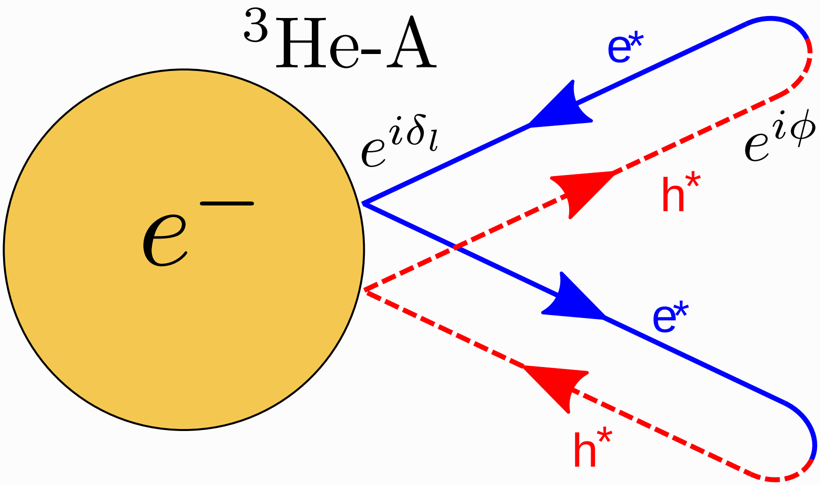
$$\frac{\eta_\perp}{\eta_{\mathsf{N}}} - 1 \propto -\Delta(T) \propto \sqrt{1 - \frac{T}{T_c}}$$

$$\frac{\eta_{\mathsf{AH}}}{\eta_{\mathsf{N}}} \propto \Delta^2(T) \propto 1 - \frac{T}{T_c}$$

- For $\dfrac{T}{T_c} \to 0^+$:

$$\frac{\eta_\perp}{\eta_{\mathsf{N}}} \propto \left(\frac{T}{T_c}\right)^2$$

$$\frac{\eta_{\mathsf{AH}}}{\eta_{\mathsf{N}}} \propto \left(\frac{T}{T_c}\right)^3$$



$$\Delta(\hat{\mathbf{k}}) = \Delta \sin\theta e^{i\phi}$$

Angular Momentum of Chiral P-wave Condensates

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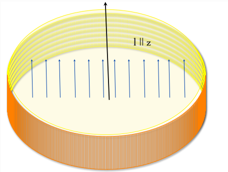
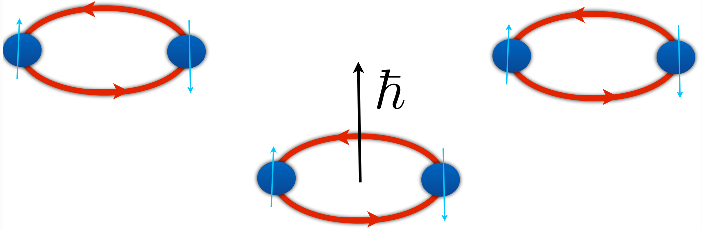
Chiral P-wave BEC Molecules or BCS Pairs (N Fermions):

$$|\,\Phi_N\,\rangle = \left[\iint d\mathbf{r}_1 d\mathbf{r}_2\, \varphi_{s_1 s_2}(\mathbf{r}_1 - \mathbf{r}_2)\, \psi^{\dagger}_{s_1}(\mathbf{r}_1)\psi^{\dagger}_{s_2}(\mathbf{r}_2)\right]^{N/2} |\,\text{vac}\,\rangle$$

- $\varphi_{s_1 s_2}(\mathbf{r}) = f(|\mathbf{r}|/\xi)\,(x+iy)\,\chi_{s_1 s_2}(S=1, M_S=0)$
  - BEC ($\xi < a$) vs. BCS ($\xi > a$)



$L_z = (N/2)\hbar$



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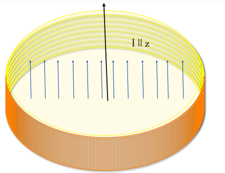

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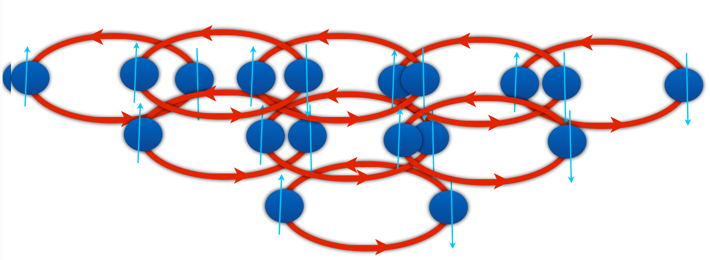

$L_z = (N/2)\hbar\,(a/\xi)^2 \ll (N/2)\hbar$? (P.W. Anderson & P. Morel, 1960, A. Leggett, 1975)

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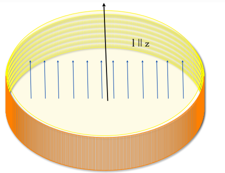

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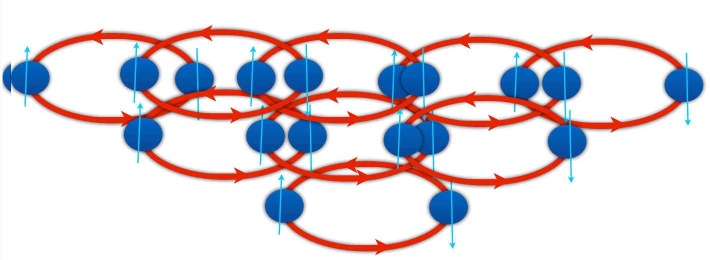

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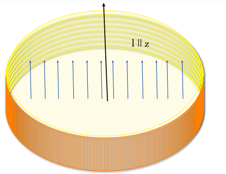

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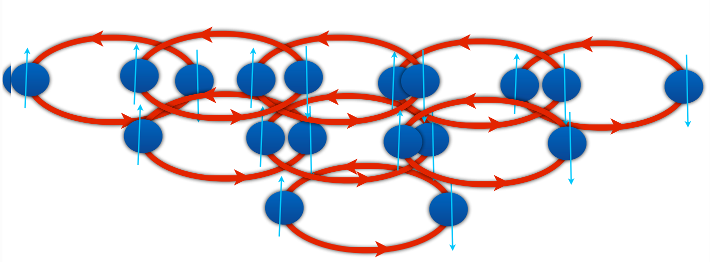

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Bogoliubov Hamiltonian for Fermions

BCS Mean Field: $\Delta_{\alpha\beta}(\mathbf{r},\mathbf{r}') = V(\mathbf{r},\mathbf{r}')\langle\psi_\alpha(\mathbf{r})\psi_\beta(\mathbf{r}')\rangle \rightsquigarrow \Delta_{\alpha\beta}(\mathbf{r},\mathbf{p})$ with $\mathbf{p} = \frac{\hbar}{i}\boldsymbol{\nabla}_{\mathbf{r}}$

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Hamiltonian:

$$\mathcal{H} = \int d\mathbf{r}\,\psi^\dagger_\alpha(\mathbf{r})\xi(\mathbf{p})\psi_\alpha(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\Big[\Delta_{\alpha\beta}(\mathbf{r},\mathbf{r}')\psi^\dagger_\alpha(\mathbf{r})\psi^\dagger_\beta(\mathbf{r}') + \Delta^*_{\alpha\beta}(\mathbf{r},\mathbf{r}')\psi_\beta(\mathbf{r}')\psi_\alpha(\mathbf{r})\Big] \tag{1}$$

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Bogoliubov Transformation: $\gamma_{n\alpha}^\dagger = \int d\mathbf{r}\left\{u_{n,\alpha}(\mathbf{r})\psi_\alpha^\dagger(\mathbf{r}) + v_{n,\alpha}^*(i\sigma_y)_{\alpha\beta}(\mathbf{r})\psi_\beta(\mathbf{r})\right\}$

$$\mathcal{H} = E_S + \sum_{n,\alpha} \varepsilon_{n,\alpha}\gamma_{n\alpha}^\dagger\gamma_{n\alpha} \quad \text{with} \quad \left\{\gamma_{n\alpha}, \gamma_{n'\beta}^\dagger\right\} = \delta_{nn'}\delta_{\alpha\beta}$$

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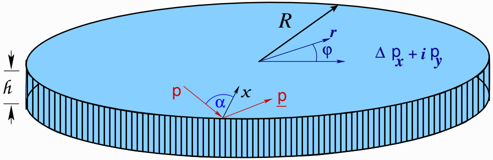
Nambu-Bogoliubov Spinors: $|\,\varphi\,\rangle = \begin{pmatrix} u \\ v \end{pmatrix}$, with $u = \begin{pmatrix} u_\uparrow \\ u_\downarrow \end{pmatrix}$, $v = \begin{pmatrix} v_\uparrow \\ v_\downarrow \end{pmatrix}$ obey

$$\begin{pmatrix} \xi(\mathbf{p})\hat{1} & \hat{\Delta}(\mathbf{r},\mathbf{p}) \\ \hat{\Delta}^\dagger(\mathbf{r},\mathbf{p}) & -\xi(\mathbf{p})\hat{1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{with} \quad \hat{\Delta}(\mathbf{r},\mathbf{p}) = i\boldsymbol{\sigma}\sigma_y \cdot \vec{\mathbf{d}}(\mathbf{r},\mathbf{p})$$

2D Chiral A-phase

^{3}He-A confined in a thin cylindrical cavity - $h \ll \xi_0$ and $R \gg \xi_0$.



- 2D Chiral ABM State: $\vec{\mathbf{d}}(\mathbf{p}) = \Delta\,\hat{\mathbf{z}}\,(p_x + ip_y)/p_f \sim e^{+i\varphi_{\mathbf{p}}}$
- Fully Gapped: $|\vec{\mathbf{d}}(\mathbf{p})|^2 = \Delta^2$

Bogoliubov Equations for Fermionic Excitations: $\mathbf{p} \rightarrow \dfrac{\hbar}{i}\boldsymbol{\nabla}$

$$\begin{pmatrix} |\mathbf{p}|^2/2m^* - \mu & \Delta\,(p_x + ip_y)/p_f \\ \Delta\,(p_x - ip_y)/p_f & -|\mathbf{p}|^2/2m^* + \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}$$

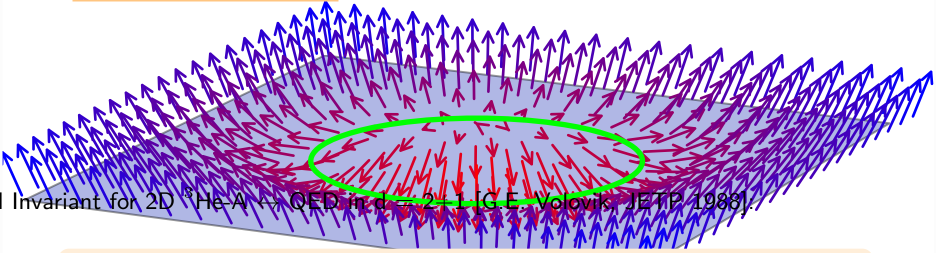
Anderson's Iso-spin Representation with particle-hole (Nambu) matrices $\widehat{\vec{\tau}} = (\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3)$

$$\widehat{H} = \left(|\mathbf{p}|^2/2m - \mu\right)\widehat{\tau}_3 + \left[\Delta\,p_x\,\widehat{\tau}_1 \mp \Delta\,p_y\,\widehat{\tau}_2\right]/p_f = \vec{\mathbf{m}}(\mathbf{p}) \cdot \widehat{\vec{\tau}}$$

Topological Invariant for 2D ^{3}He-A and Fermionic Spectrum

Nambu-Bogoliubov Hamiltonian for 2D ^{3}He-A : $\widehat{H} = \vec{\mathbf{m}}(\mathbf{p}) \cdot \widehat{\vec{\boldsymbol{\tau}}}$

$\rightsquigarrow$ $\vec{\mathbf{m}} = (\, cp_x \,,\, \mp cp_y \,, \xi(\mathbf{p}))$ with $|\vec{\mathbf{m}}(\mathbf{p})|^2 = \left(|\mathbf{p}|^2/2m - \mu\right)^2 + c^2|\mathbf{p}|^2 > 0 \,, \mu \neq 0$



Topological Invariant for 2D ^{3}He-A $\leftrightarrow$ QED in d = 2+1 [G.E. Volovik, JETP 1988]:

$$N_{\text{2D}} = \pi \int \frac{d^2p}{(2\pi)^2}\, \hat{\mathbf{m}}(\mathbf{p}) \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial p_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial p_y} \right) = \begin{cases} \pm 1\,; & \mu > 0 \text{ and } \Delta \neq 0 \\ 0\,; & \mu < 0 \text{ or } \Delta = 0 \end{cases}$$

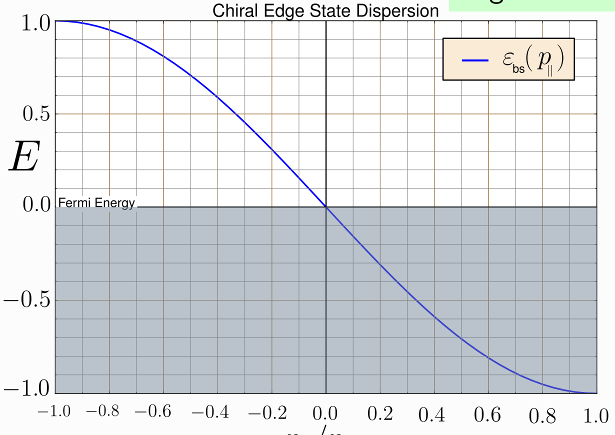
"Vacuum" $(\Delta = 0)$ with $N_{2D} = 0$ | ^{3}He-A$(\Delta \neq 0)$ with $N_{2D} = 1$

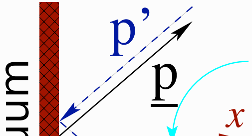
Zero Energy Fermions ↑ Confined on the Edge

Edge Fermions: $G^{\text{R}}_{\text{edge}}(\mathbf{p},\varepsilon;x) = \dfrac{\pi\Delta|\mathbf{p}_x|}{\varepsilon + i\gamma - \varepsilon_{\text{bs}}(\mathbf{p}_{||})}\, e^{-x/\xi_\Delta}$

Confinement: $\xi_\Delta = \hbar v_f/2\Delta \approx 10^2 - 10^3\ \mathring{A} \gg \hbar/p_f$

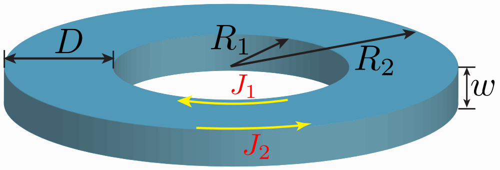
- $\varepsilon_{\text{bs}} = -c\, p_{||}$ with $c = \Delta/p_f \ll v_f$
- Broken P & T $\rightsquigarrow$ Edge Current





Ground-State Angular Momentum of ^{3}He-A in a Toroidal Geometry

^{3}He-A confined in a toroidal cavity



- $R_1, R_2, R_1 - R_2 \gg \xi_0$

- Sheet Current: $J = \frac{1}{4}\, n\, \hbar$ ($n = N/V =$ ^{3}He density)
- Counter-propagating Edge Currents: $J_1 = -J_2 = \frac{1}{4}\, n\, \hbar$
- Angular Momentum:

$$L_z = 2\pi\, h\, (R_1^2 - R_2^2) \times \frac{1}{4}\, n\, \hbar = (N/2)\, \hbar$$

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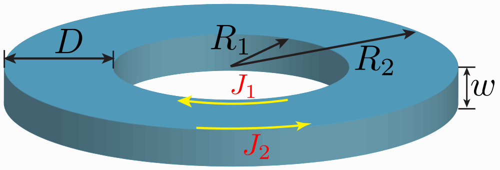

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McClure-Takagi's Global Symmetry Result PRL 43, 596 (1979)

Chiral Edge Currents

Local Density of States: $N(\mathbf{p}, x; \varepsilon) = -\frac{1}{\pi} \, \text{Im} \, \mathfrak{g}^{\text{R}}(\mathbf{p}, x; \varepsilon)$

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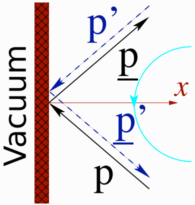
Pair Time-Reversed Trajectories

Spectral Current Density :

$$\vec{J}(\mathbf{p}, x; \varepsilon) = 2N_f \, \mathbf{v}(\mathbf{p}) \left[N(\mathbf{p}, x; \varepsilon) - N(\mathbf{p}', x; \varepsilon) \right]$$



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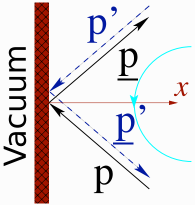

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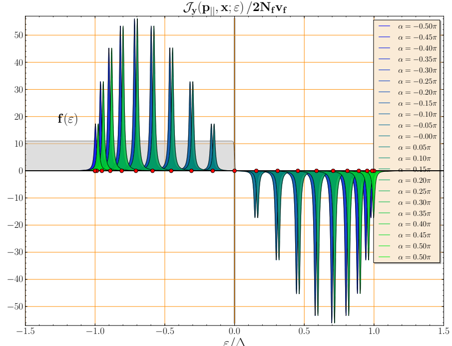
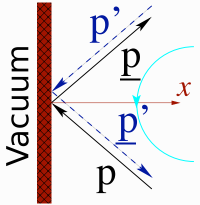
Bound-State Edge Current at $x = 0$



Chiral Edge Currents

Local Density of States: $N(\mathbf{p},x;\varepsilon) = -\frac{1}{\pi}\,\text{Im}\,\mathfrak{g}^{\text{R}}(\mathbf{p},x;\varepsilon)$

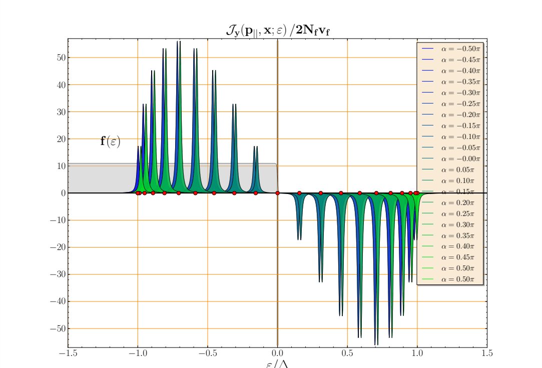


Pair Time-Reversed Trajectories

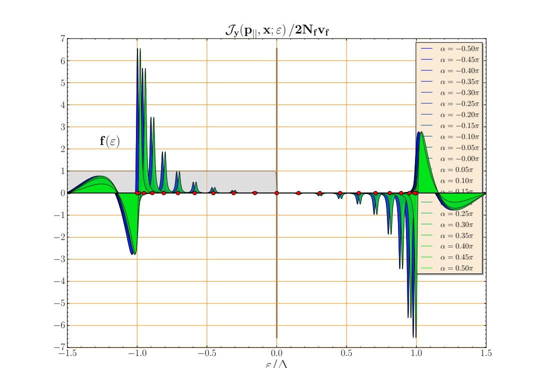
Spectral Current Density :

$$\vec{J}(\mathbf{p},x;\varepsilon) = 2N_f\,\mathbf{v}(\mathbf{p})\left[N(\mathbf{p},x;\varepsilon) - N(\mathbf{p}',x;\varepsilon)\right]$$

Bound-State Edge Current at $x = 0$

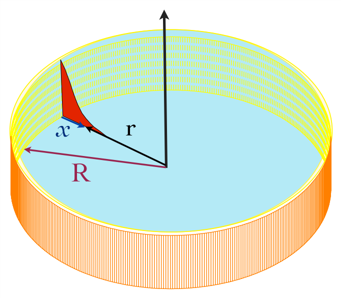


Continuum Edge Current at $x = 10\xi_0$



Edge Currents and Angular Momentum

Ground-State Current Density: $\vec{J}(x) = \int_{-p_f}^{+p_f} \frac{dp_{||}}{p_f} \int_{-\infty}^{0} \vec{J}(\mathbf{p}, x; \varepsilon)$



Bound-State Contribution ($R \gg \xi_\Delta$):

$$
\begin{aligned}
J_\varphi(\mathbf{p}, x; \varepsilon) &= 2N_f\, v_f\, \Delta\, |p_x|\, p_\varphi\, e^{-x/\xi_\Delta} \\
&\times \left[\delta(\varepsilon - \varepsilon_{\mathsf{bs}}(\mathbf{p}_{||})) - \delta(\varepsilon - \varepsilon_{\mathsf{bs}}(\mathbf{p'}_{||}))\right]
\end{aligned}
$$

Bound-State Edge Current: $\int_0^\infty dx J_\varphi(x) = \frac{1}{2}\, n\, \hbar$

Mass Current: $v_f \to p_f \rightsquigarrow \vec{J} \to \vec{g}$

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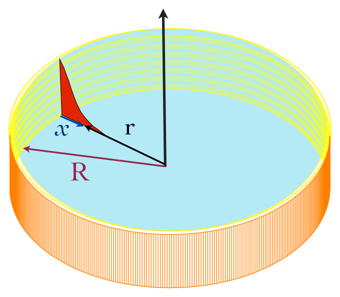

Bound-State Contribution ($R \gg \xi_\Delta$):

$$J_\varphi(\mathbf{p}, x; \varepsilon) = 2N_f\, v_f\, \Delta\, |p_x|\, p_\varphi\, e^{-x/\xi_\Delta} \times \left[\delta(\varepsilon - \varepsilon_{\text{bs}}(\mathbf{p}_{||})) - \delta(\varepsilon - \varepsilon_{\text{bs}}(\mathbf{p'}_{||}))\right]$$

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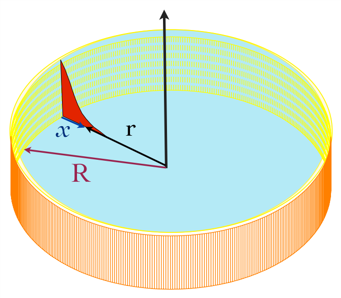

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- Continuum ($\varepsilon < -\Delta$): $J_\varphi^{\text{C}} = 2N_f\, v_f\, |p_x| \left(\frac{\Delta^2\, p_\varphi^2}{\varepsilon^2 - \varepsilon_{\text{bs}}^2(\mathbf{p}_{||})}\right) \sin\left(2\sqrt{\varepsilon^2 - \Delta^2}\, x/v_x\right)$

Edge Currents and Angular Momentum

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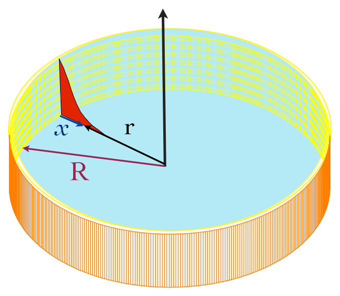

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$$J_\varphi(\mathbf{p}, x; \varepsilon) = 2N_f\, v_f\, \Delta\, |p_x|\, p_\varphi\, e^{-x/\xi_\Delta} \times \left[\delta(\varepsilon - \varepsilon_{\mathsf{bs}}(\mathbf{p}_{||})) - \delta(\varepsilon - \varepsilon_{\mathsf{bs}}(\mathbf{p}'_{||}))\right]$$

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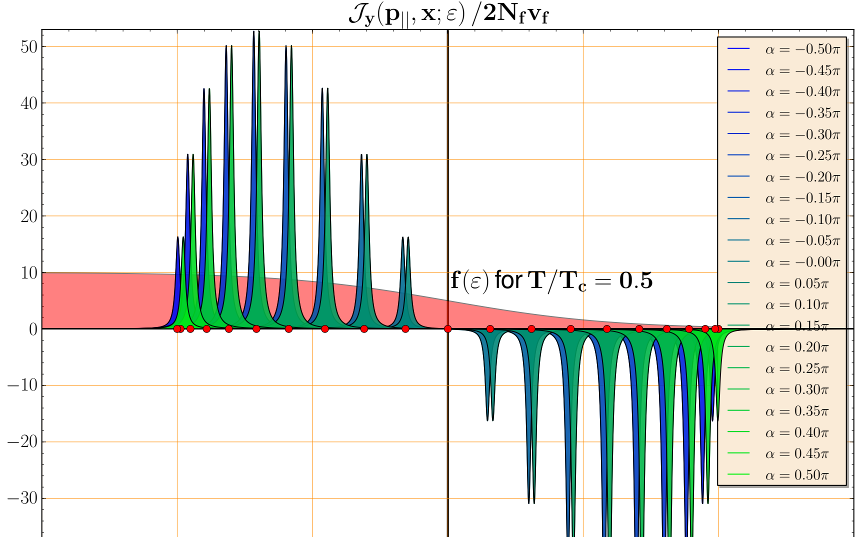
- $L_z^{\mathsf{C}} = \int_V d^2r\; \left[r\, g_\varphi^{\mathsf{C}}(\mathbf{r})\right] = -\frac{1}{2} N\,\hbar \rightsquigarrow L_z^{\mathsf{total}} = (N/2)\hbar$ - MT Result Recovered!

Thermal Excitation of Chiral Edge Fermions

Thermally Excited Edge Fermions Carry the Opposite Current

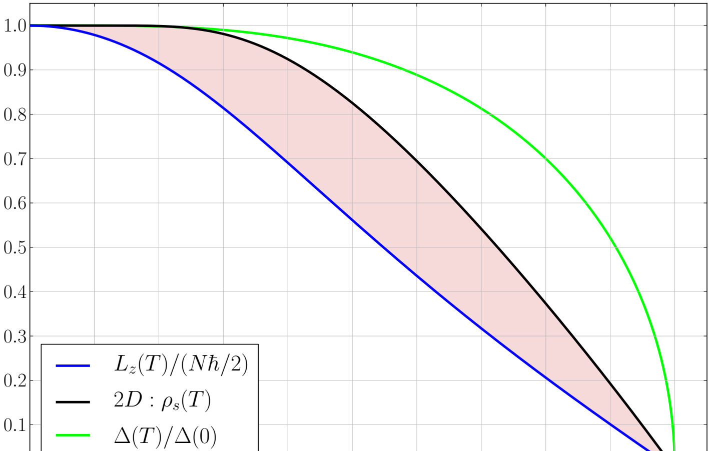


Angular Momentum of ^{3}He-A vs. Temperature

$$J = \frac{1}{4}\, n\, \hbar \times \mathcal{Y}_{\text{edge}}(T) \qquad \mathcal{Y}_{\text{edge}}(T) \approx 1 - c\,(T/\Delta)^2\,, \quad T \ll \Delta$$

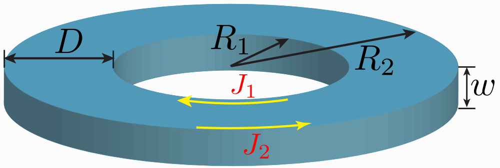
- Thermal Signature of the Chiral Edge States

$$\rho_s(T)/\rho = \mathcal{Y}_{\text{bulk}}(T) - 1 \propto -\, e^{-\Delta/T}\,, \quad T \ll \Delta$$



Ground-State Angular Momentum of ^{3}He-A in a Toroidal Geometry

^{3}He-A confined in a toroidal cavity



- $R_1, R_2, R_1 - R_2 \gg \xi_0$

- Sheet Current: $J = \frac{1}{4}\, n\, \hbar$ ($n = N/V = {}^3$He density)
- Counter-propagating Edge Currents: $J_1 = -J_2 = \frac{1}{4}\, n\, \hbar$
- Angular Momentum:

$$L_z = 2\pi\, h\, (R_1^2 - R_2^2) \times \frac{1}{4}\, n\, \hbar = (N/2)\,\hbar$$

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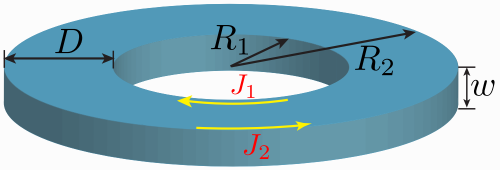

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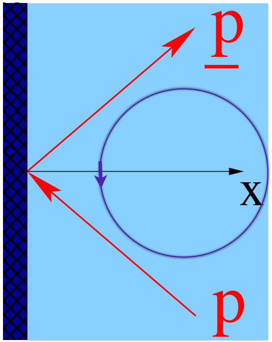
McClure-Takagi's Global Symmetry Result PRL 43, 596 (1979)

Magnitude of Edge Currents are Protected by Symmetry

Magnitude of Edge Currents are Protected by Symmetry

Specular Reflection



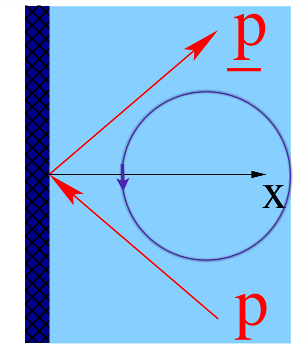
Propagating Chiral Fermions:

$$\mathfrak{g}^{\mathsf{R}}(\mathbf{p},\varepsilon;x) = \frac{\pi\Delta|\mathbf{p}_x|}{\varepsilon + i\gamma - \varepsilon_{\mathsf{bs}}(\mathbf{p}_{||})}\, e^{-x/\xi_\Delta}$$

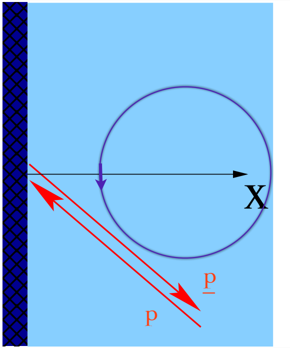
Edge Current: $J = \frac{1}{4}\, n\, \hbar$

Magnitude of Edge Currents are Protected by Symmetry

Specular Reflection



Retro Reflection



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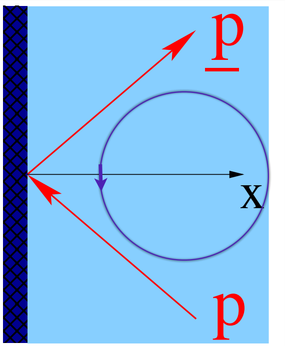

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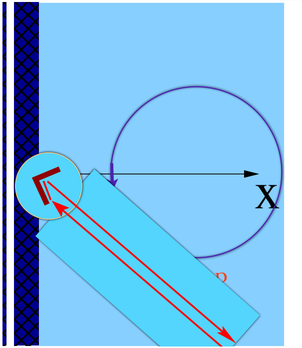

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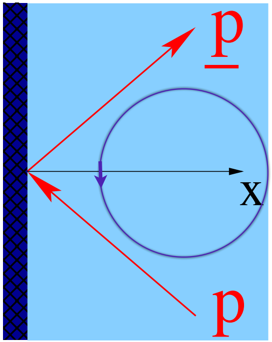

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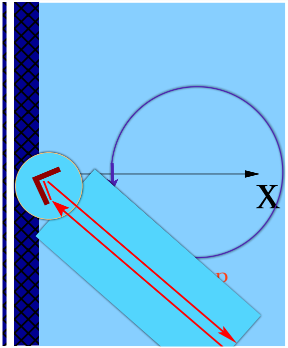
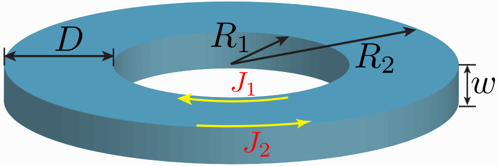


Zero-Energy Fermions for all $\mathbf{p}$:

$$\mathfrak{g}^{\mathsf{R}}(\mathbf{p},\varepsilon;x) = \frac{\pi\Delta}{\varepsilon + i\gamma}\, e^{-2\Delta x/v_x}$$

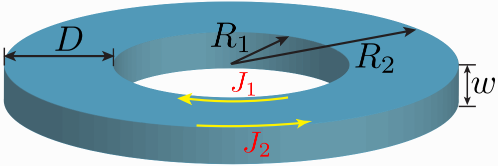
$\rightsquigarrow$ Edge Current: $J = 0$

Magnitude of Edge Currents are Protected by Symmetry



- Sheet Current: $J = f \times \frac{1}{4} n \hbar$
- Non-Specular Surfaces
  $0 \leq f \leq 1$

Magnitude of Edge Currents are Protected by Symmetry



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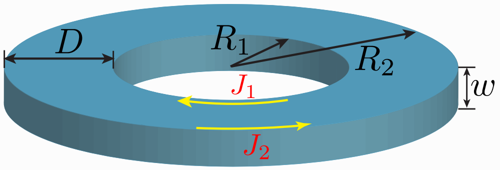
Incomplete Screening of Counter-Propagating Currents

Scaling of L_z with $r = (R_1/R_2)^2$ $\quad 0 < r < 1$

Magnitude of Edge Currents are Protected by Symmetry



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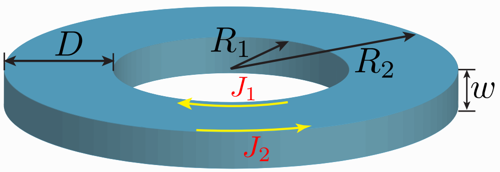
- $f_1 = 1,\ f_2 = 0$

$$L_z = (N/2)\,\hbar \times \left(\frac{1}{1-r}\right) \gg (N/2)\,\hbar$$

Magnitude of Edge Currents are Protected by Symmetry



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Scaling of L_z with $r = (R_1/R_2)^2 \qquad 0 < r < 1$

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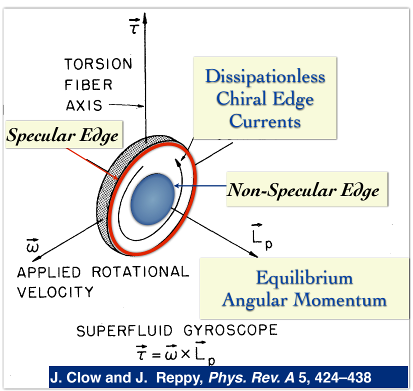
  $$L_z = (N/2)\,\hbar \times \left(\frac{1}{1-r}\right) \gg (N/2)\,\hbar$$

- $f_1 = 0,\ f_2 = 1$

  $$L_z = (N/2)\,\hbar \times \left(\frac{-r}{1-r}\right) \ll -(N/2)\,\hbar$$

- Strong violations of the McClure-Takagi Result

Gyroscopic Experiment to Measure of $L_z(T)$



Thermal Signature of Chiral Edge States

- Power Law for $T \lesssim 0.5 T_c$

$$L_z = (N/2)\hbar\,(1 - c\,(T/\Delta)^2)$$

Toroidal Geometry with Engineered Surfaces

- Incomplete Screening

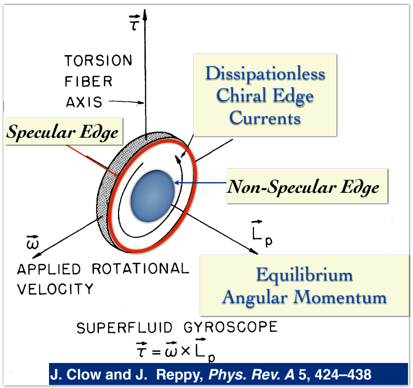
$$L_z > (N/2)\hbar$$

Direct Signature of Edge Currents

Long-Standing Challenge: The Ground-State Angular Momentum of ^{3}He-A

Possible Gyroscopic Experiment to Measure of $L_z(T)$

- Hyoungsoon Choi (KAIST) [sub-micron mechanical gyroscope @ 200 μK]



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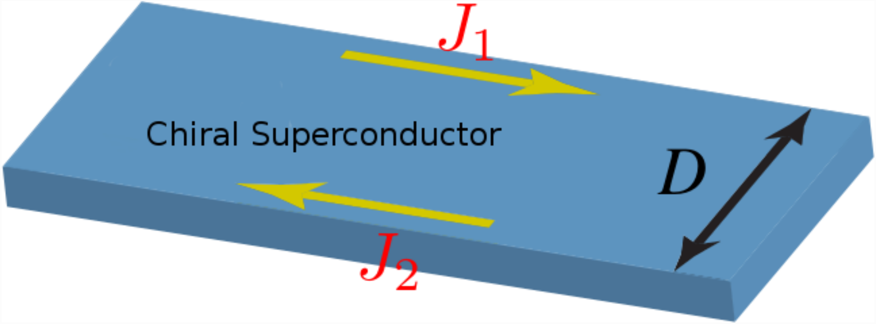
Direct Signature of Edge Currents

- J. A. Sauls, Phys. Rev. B 84, 214509 (2011)
- Y. Tsutsumi, K. Machida, JPSJ 81, 074607 (2012)

- Mesoscopic geometries: Edge states are important for transport



- Surface states, edge currents, and the angular momentum of chiral p-wave superfluids and superconductors, JAS, Phys. Rev. B 84, 214509 (2011) [arXiv:1209.5501]

- Symmetry Protected Topological Superfluids and Superconductors — From the Basics to ^{3}He, T. Mizushima, Y. Tsutsumi, T. Kawakami, M. Sato, M. Ichioka, K. Machida [arXiv:1508.00787]